

SIGNALS AND SYSTEMS LABORATORY 2:

Phasors, Interference, Beating Between Tones, and Lissajous Figures

INTRODUCTION

“In engineering and applied science, three test signals form the basis for our study of electrical and mechanical systems. The *impulse* is an idealized signal that models very short excitations (like current pulses, hammer blows, pile drives, and light flashes). The *step* is an idealized signal that models excitations that are switched on and stay on (like current in a relay that closes or a transistor that switches). The *sinusoid* is an idealized signal that models excitations that oscillate with a regular frequency (like AC power, AM radio, pure musical tones, and harmonic vibrations). All three signals are used in the laboratory to design and analyze electrical and mechanical circuits, control systems, radio antennas, and the like. The sinusoidal signal is particularly important because it may be used to determine the frequency selectivity of a circuit (like a superheterodyne radio receiver) to excitations of different frequencies. For this reason, every manufacturer of electronic test equipment builds sinusoidal oscillators that may be swept through many octaves of frequency. (Hewlett-Packard was started in 1940 with the famous HP audio oscillator.)”
-- taken from “A First Course in Electrical and Computer Engineering with MATLAB Programs and Experiments”, by L. L. Scharf and R. T. Behrens, Addison-Wesley, 1990.

PHASORS

Consider the co-sinusoidal signal $x(t) = A \cos(\omega t + \theta)$, with amplitude A in units of volts, frequency ω in units of rad/sec, and phase θ in units of radians.

Assignment:

1. Write $x(t)$ as $A \cos(2\pi f t + \theta)$ and $A \cos(2\pi f (t - \tau))$. Find f and τ and give their units.

The *real* signal $x(t)$ may be written as the real part of a *complex* signal:

$$x(t) = \operatorname{Re}\{z(t)\}$$
$$z(t) = A e^{j(\omega t + \theta)} = A e^{j\theta} e^{j\omega t}$$

The complex signal $z(t)$ is a *rotating* phasor consisting of the *stationary* phasor $A e^{j\theta}$, rotated by $e^{j\omega t}$. In summary, $x(t) = \operatorname{Re}\{A e^{j\theta} e^{j\omega t}\}$. As the next paragraph illustrates, we may perform many important calculations by ignoring the rotator $e^{j\omega t}$ and considering only the stationary phasor $A e^{j\theta}$.

Assignment:

- 2.) Run demo ‘[rotating_phasor.m](#)’ under ‘Demos for Lab 2’ to see how the phasor $10e^{-j\pi/4}$ rotates by $e^{j\omega t}$ at uniform angular rate of $\omega = 2\pi 10^3$ rad/sec. Change the amplitude, phase, and frequency at will to demonstrate the action of a rotating phasor. Explain what each parameter does.

INTERFERENCE

When two, or more, signals are received at a linear sensor or antenna, they will add. Depending on the relative phase, or time alignment, of the signals, they may add constructively or destructively. A fading radio transmission is a consequence of destructive interference of signals that are misaligned in time by virtue of multi-path propagation. A large gain in a filter is a consequence of constructive interference of signals that are aligned in time.

Lets elaborate by considering the addition, or interference, of two signals of the same amplitude and frequency, but different phases (or delays):

$$x(t) = A \cos(\omega t + \theta_1) + A \cos(\omega t + \theta_2)$$

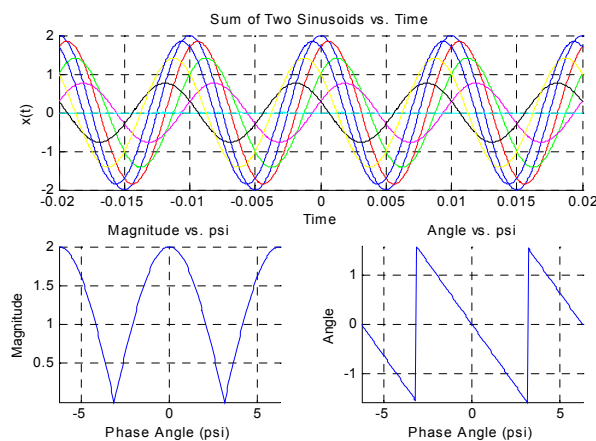
Now watch carefully as we use phasor calculus to simplify the effect of interference:

$$\begin{aligned} x(t) &= \operatorname{Re}\{A e^{j\theta_1} e^{j\omega t}\} + \operatorname{Re}\{A e^{j\theta_2} e^{j\omega t}\} \\ &= \operatorname{Re}\{(A e^{j\theta_1} + A e^{j\theta_2}) e^{j\omega t}\} \\ &= \operatorname{Re}\left\{A e^{j\frac{\theta_1+\theta_2}{2}} \left(e^{j\frac{\theta_1-\theta_2}{2}} + e^{j\frac{-(\theta_1-\theta_2)}{2}} \right) e^{j\omega t}\right\} \\ &= \operatorname{Re}\{A e^{j\theta} 2 \cos(\psi/2) e^{j\omega t}\} \\ &= 2A \cos(\psi/2) \cos(\omega t + \theta), \end{aligned}$$

In this formula, $\theta = \frac{\theta_1 + \theta_2}{2}$ is the average phase, and $\psi = \theta_1 - \theta_2$ is the phase difference. The interference effect is this: when $\psi = 0$, then $\cos(\psi/2) = 1$ and the output, $x(t)$, is doubled in amplitude. For $\psi = \pi$, then $\cos(\psi/2) = 0$ and the output is zeroed. For in-between values of the phase difference, the gain is between 0 and 2.

Assignment:

- The function $2A \cos(\psi/2)$ is called the interference pattern for two signals of equal amplitude and frequency, and phase difference ψ . Run the demo '[interference_pattern.m](#)' under 'Demos for Lab 2' to plot the interference pattern. You should see something like this:



Explain what you see. Pay particular attention to the bottom two graphs, which are the magnitude and phase of the complex interference pattern. For your own amusement, you might want to write an interference program that accepts signals of different amplitudes.

Assignment:

4. Now generalize the previous development by considering the interference of these N signals:

$$x(t) = A \cos(\omega t) + A \cos(\omega t + \theta) + A \cos(\omega t + 2\theta) + \dots + A \cos(\omega t + (N-1)\theta)$$

Show that this may be written as

$$\begin{aligned} x(t) &= \operatorname{Re} \left\{ A \frac{\sin(N\theta/2)}{\sin(\theta/2)} e^{j \frac{N-1}{2} \theta} e^{j\omega t} \right\} \\ &= A \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cos\left(\omega t + \frac{N-1}{2} \theta\right) \end{aligned}$$

Rewrite the interference program '[interference_pattern.m](#)' to plot your newly derived interference

pattern, $\frac{\sin(N\theta/2)}{\sin(\theta/2)} e^{j \frac{N-1}{2} \theta}$, vs. θ . Explain what you see. What can you say about $\lim_{\theta \rightarrow 0} \frac{\sin(N\theta/2)}{\sin(\theta/2)}$?

BEATING BETWEEN TONES

In the section on interference, we dealt with signals that had the same amplitude and angular frequency, but different phases. Now, we are going to allow the angular frequencies to differ and study the phenomenon of beating. As with the interference experiment, we are going to assume that the amplitudes of the signals are equal. Then the sum of two signals with different frequencies is equal to $y(t) = A \cos(\omega_1 t) + A \cos(\omega_2 t)$.

Assignment:

5. Run the demo '[beating_tones.m](#)' from the class webpage, under 'Demos for Lab 2', to plot $y(t)$ vs. t .

Experiment with various choices of ω_1 and ω_2 to control the beat frequency $\left(\frac{\omega_1 - \omega_2}{2}\right)$ and the

average frequency $\left(\frac{\omega_1 + \omega_2}{2}\right)$.

Show that $y(t) = A \cos(\omega_1 t) + A \cos(\omega_2 t)$ may be written as a cosine at average frequency

$\left(\frac{\omega_1 + \omega_2}{2}\right)$, multiplied by a cosine at the beat frequency $\left(\frac{\omega_1 - \omega_2}{2}\right)$:

$$y(t) = 2A \cos\left[\left(\frac{\omega_1 - \omega_2}{2}\right)t\right] \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right)t\right]$$

LISSAJOUS FIGURES

A Lissajous figure is a figure that is displayed on an oscilloscope in the 'xy' mode. We can simulate these figures by graphing the equation $z(t) = A_1 \cos(\omega t + \phi_1) + jA_2 \cos(\omega t + \phi_2)$ on the complex plane.

Assignment:

6. Download and run the program 'lissajous.m' from the class web page under 'Demos for Lab 2'. Show that a Lissajous figure is just the sum of two rotating phasors, one of which rotates counter-clockwise at angular frequency ω and the other which rotates clockwise at angular frequency ω . When is the Lissajous figure of the form $Ae^{j\theta}e^{j\omega t}$, and what kind of figure is this?

END NOTES:

The following m-files are located on the web page under 'Lab 2'. These files are for your benefit. Please do not turn in any of the results from these exercises or demos.

To get a better understanding of phasors, try the following extra exercises:

Damped Phasor: ['damped_phasor.m'](#)

For your own amusement, try the following extra demos:

Bicycle Wheel: ['bicycle_wheel.m'](#)

Spirograph: ['spirograph.m'](#)

Maypole: ['maypole.m'](#)

Curves Of Unity: ['curves_of_unity.m'](#)