Note on Circular vs Linear Convolution.

We know that a sampled sequence $x[n] = x(nt_0)$ has a related representation in the frequency domain

$$x[n] \longleftrightarrow X(z)\Big|_{z=e^{j\omega}} = X(\omega)$$

$X(\omega)$ is the DTFT (not DFT). That is,

$$x[n] = \mathcal{F}T^{-1}\{X(\omega)\}.$$

We can recover $x[n]$ by doing an inverse Fourier Transform on $X(\omega)$.

According to the convolution theorem,

$$x[n] \circledast y[n] \longleftrightarrow X(\omega) \cdot Y(\omega)$$

However, if instead we use the DFT to represent $x[n]$ and $y[n]$, this is not true:

$$x[n] \circledast y[n] \not\longleftrightarrow X[k] \cdot Y[k]$$

Instead, the following holds:

$$x_p[n] \circledast y_p[n] \longleftrightarrow X[k] \cdot Y[k]$$

where x_p and y_p are the periodically-extended versions of $x[n]$ and $y[n]$.

Instead of using the periodically-extended versions, we can use an operation called <u>circular convolution</u> of the original sequences.

The following is true:

$$x[n] \; \textcircled{N} \; y[n] \longleftrightarrow X[k] \cdot Y[k]$$

The $\textcircled{N}$ symbol signifies circular convolution over length N, where N is the order of the DFTs.

The following defines circular convolution:

$$x[n] \; \textcircled{N} \; y[n] = \sum_{k=0}^{N-1} x[k] \, y[(n-k)_N]$$

where $(n-k)_N$ means that integer multiples of N are added or subtracted if $(n-k)$ is outside $0 \ldots N-1$:

$$(n-k)_N = \begin{cases} n-k+mN & \text{for } (n-k) \neq 0 \cdots N-1 \\ n-k & \text{for } (n-k) = 0 \cdots N-1 \end{cases}$$

If, however, we pad both sequences with 0's to length N, where N satisfies

$N \geq M+L-1$, M, L are the lengths of $x[n]$ and $y[n]$,

then circular and linear ("ordinary") convolution will give the same results over the range $0 \leq n < N$, assuming $x[n]$ and $y[n]$ have support (non-zero region) starting at $n=0$.

To summarize,

$$x[n] \; \textcircled{N} \; y[n] \equiv x_p[n] \circledast y_p[n]$$

where $x_p[n+N] = x_p[n], \ldots$