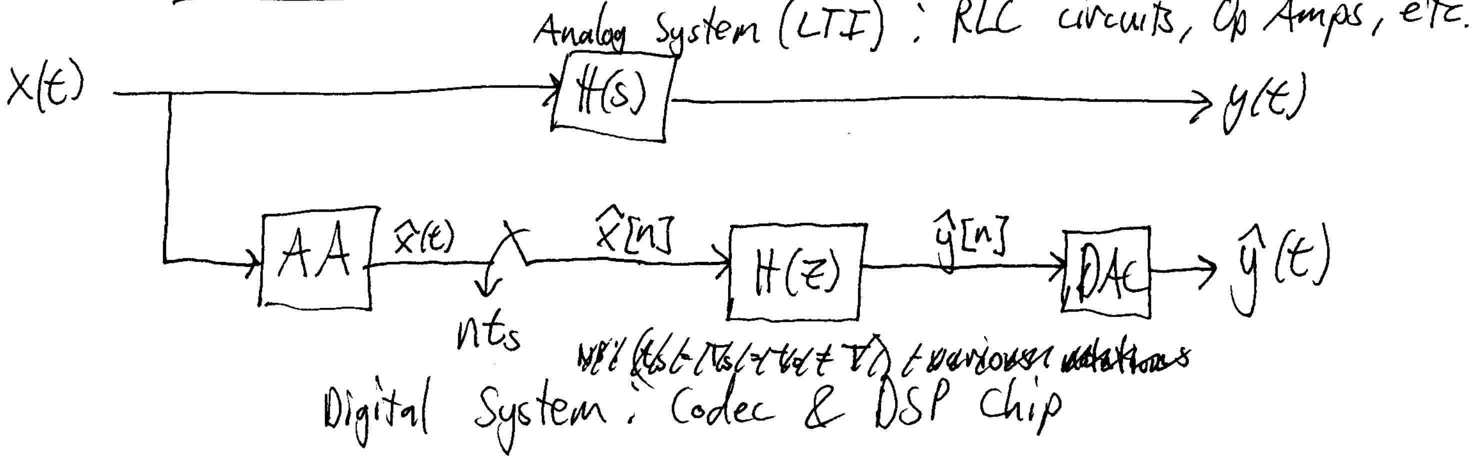
Lecture 3 : Sampling & Reconstruction
(2 50min lect.)

The DSP Idea



Notes: $\wedge \equiv$ some information about the signal has been lost

If $H(z) = 1$ (or $H(s) = 1$) then this system implements a "straight wire". (i.e. $y(t) = x(t)$)

Question : Under what conditions does $\hat{y}(t) \cong y(t)$?

To answer this, let's examine the above diagram in more detail.

~~Analog to Digital (Coding) on the DSK~~

~~Input $x(t) \xleftrightarrow{FT} X(j\omega)$ or $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ and $X(j\omega) = X(s)\big|_{s=j\omega}$~~

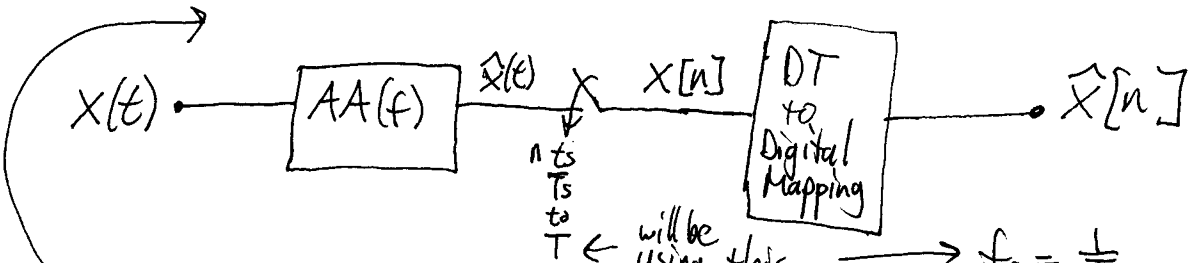
~~In order to process the signal digitally, the source must be bandlimited in frequency.~~

The Modulator: Analog to Digital Conversion (Coding)

Analog Signal:

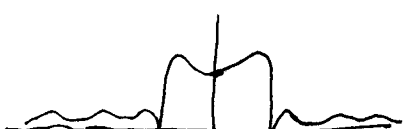
Discrete-Time Signal:

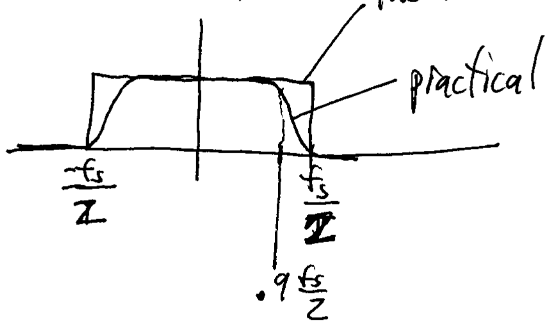
Digital Signal:



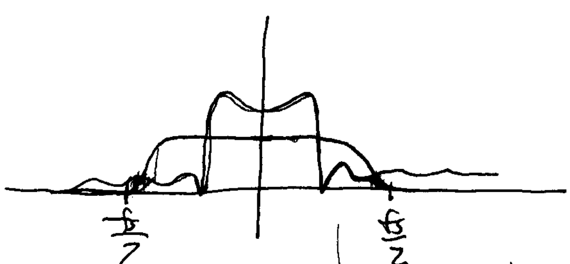
NB: $\underset{\text{rad/sec}}{AA(j\omega)} \equiv AA(j2\pi \underset{\text{Hz}}{f}) \equiv AA(\underset{\text{Hz}}{f})$

$x(t) \longleftrightarrow X(f)$ $\qquad\qquad$ $AA(f)$





$\hat{x}(t) \longleftrightarrow \hat{X}(f)$



→ some information is lost

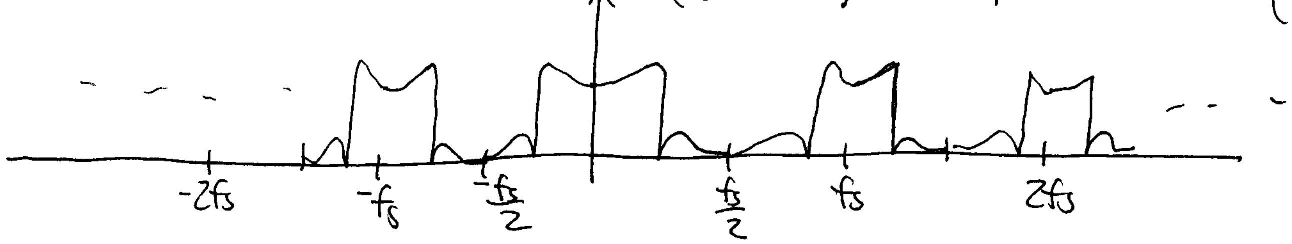
$$x[n] = \hat{x}(nT)$$

$\updownarrow$

$$X(e^{j2\pi fT}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{X}(f - kT)$$ (Sampling Theorem)

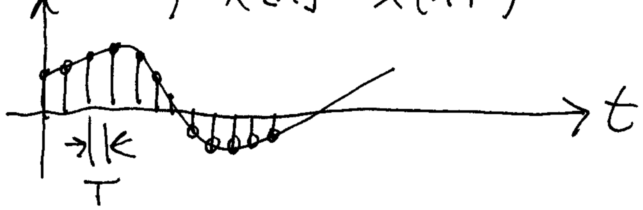
integers $m \in \mathbb{Z}$

$X(e^{j2\pi fT})$: periodic $f = m\left(\frac{1}{T}\right) = mf_s$

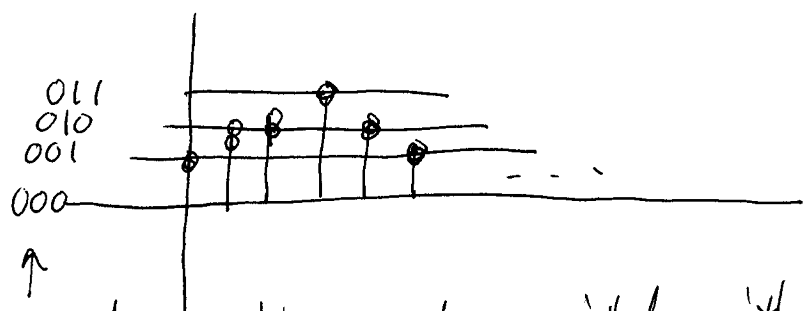


Without the AA filter, these spectra would overlap, which results in aliasing.

$\hat{x}(t)$, $x[n] = \hat{x}(nT)$

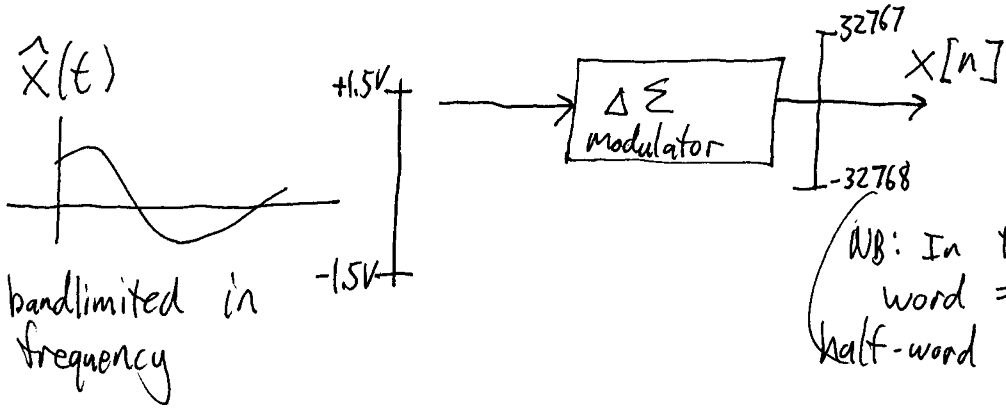


In order to represent these samples as a binary word, the discrete-time samples must be quantized to specific amplitude intervals and assigned a binary value based on quantized amplitude.



↑ corresponding 3 bit word associated with each digital sample.

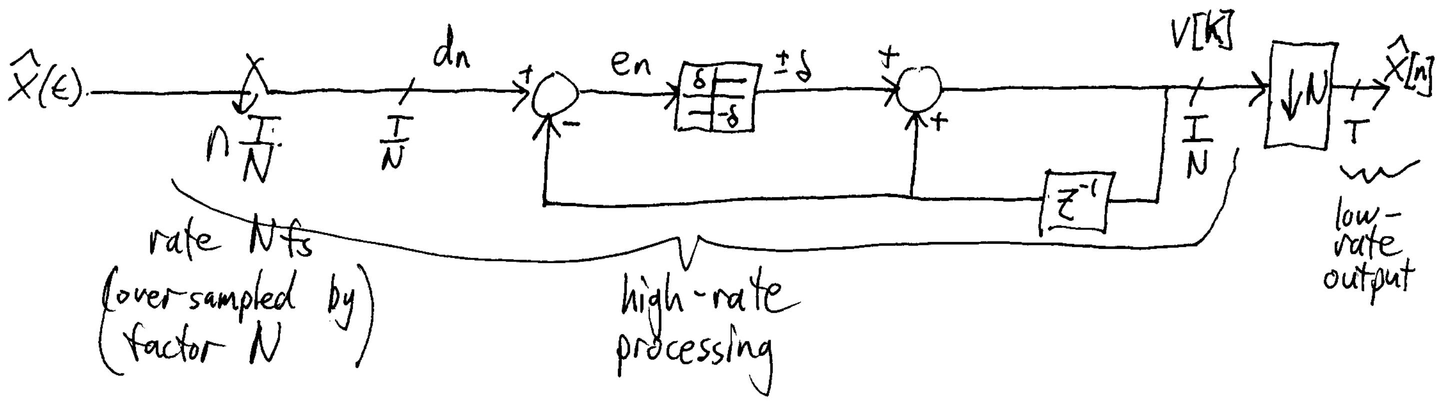
On the DSK



16-bit finite-length words @ rate f_s

(NB: In the TI literature
word = 32-bit word
half-word = 16-bit word)

The Idea (not a standard model)



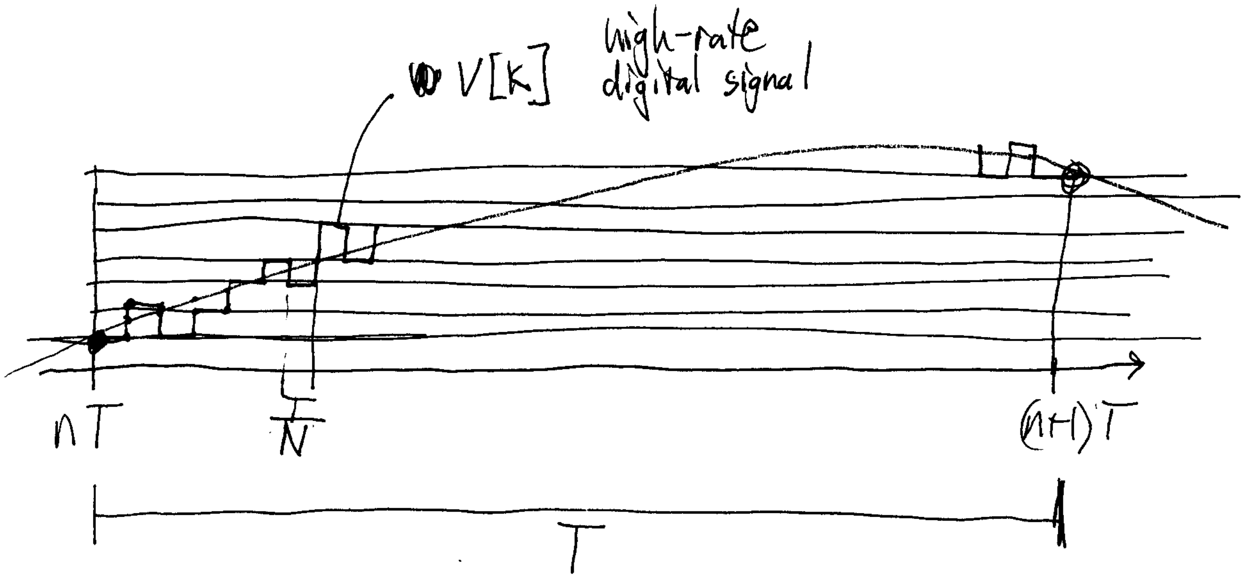
d_n, e_n : DT measurement

$v[k]$: high-rate digital signal

$\hat{x}[n]$: low-rate digital signal output

$\hat{x}[n] = v\left[\frac{k}{N}\right]$ for $\frac{k}{N}$ an integer

not intuitive - see picture



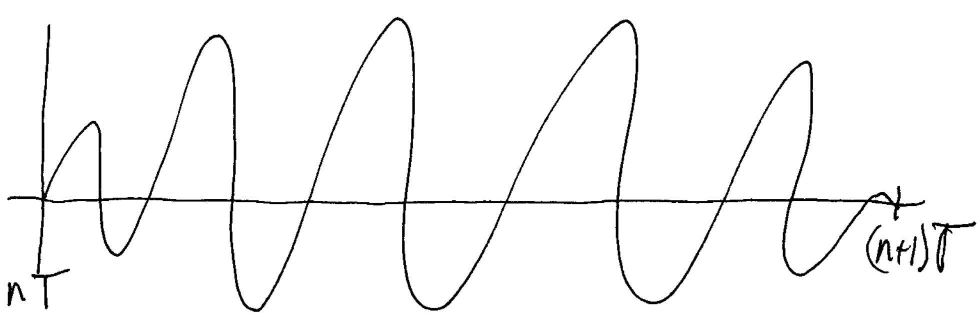
$$x[n] = x[n-1] + \delta_1 + \delta_2 + \dots + \delta_{N-1}$$

$$= x[n-1] + \underbrace{\sum_{\ell=1}^{N-1} \delta_\ell}$$

sum of correction

sigma delta

Can we have?



On the DSK (standard),

δ = one bit correction
$N = 64$

$$V[k] \xrightarrow[\frac{T}{N}]{} \boxed{\downarrow N} \xrightarrow[T]{} x[n]$$

Digital to Analog Conversion (Decoding)

Basic idea (theoretical)

Notation
$T = t_s$ = sample duration
$\frac{1}{T} = f_s$ = sampling frequency

$$y[n] \longrightarrow \boxed{H(s)} \longrightarrow z(t)$$

D/A

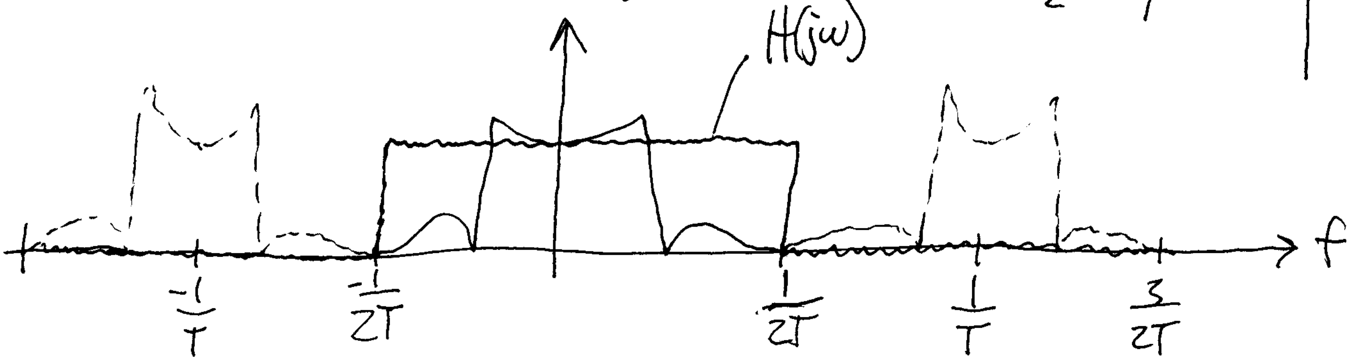
We could do this with a really good analog reconstruction filter

$$y[n] \longleftrightarrow Y(e^{j2\pi fT})$$

Notation

rad/sec	Hz
ω	f
$\frac{\omega_s}{2} = \frac{\pi}{T}$	$\frac{1}{2T} = \frac{f_s}{2}$

$Y(e^{j2\pi fT})$



$$H(j\omega) = \begin{cases} 1 & |\omega| \le \frac{1}{2T} \\ 0 & |\omega| > \frac{1}{2T} \end{cases} \longleftrightarrow h(t) = \text{sinc}\left(\frac{\pi t}{T}\right)$$

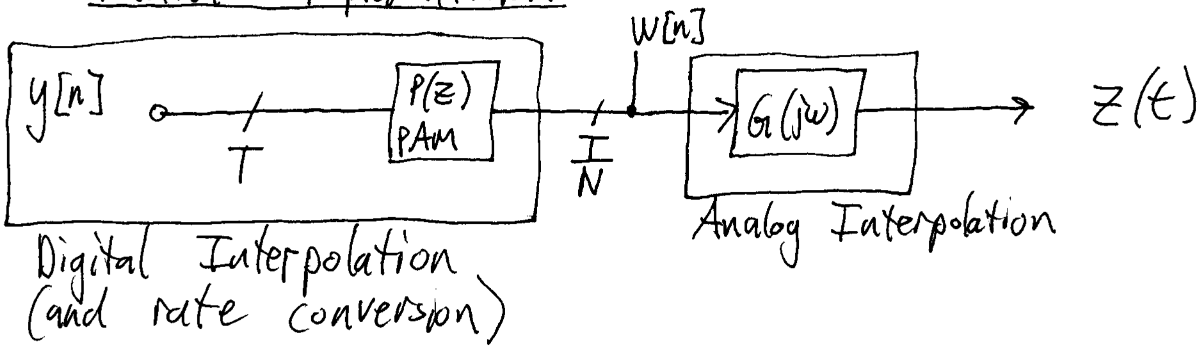
$$z(t) = \sum_{k=-\infty}^{\infty} y[k]\, h(t - kt_0) \longleftrightarrow Z(j\omega) = H(j\omega)\, Y(e^{j2\pi fT})$$

(see Sig & Sys Lab 10 Figure 3 for a time domain plot of this)

<u>Problems</u>: Building $H(j\omega)$ is impossible.

Approximating $H(j\omega)$ is expensive (no room for roll-off)

Practical implementation



Comments: $y[n]$: digital signal @ rate $\frac{1}{T}$ samples/sec

$w[n]$: digital signal @ rate $\frac{N}{T}$ samples/sec.

$z(t)$: reconstructed analog signal that is (approximately) bandlimited to $\frac{1}{2T}$ Hz.

$y[n] \longleftrightarrow Y(e^{j2\pi fT})$ – periodic in frequency with period $\frac{1}{T}$ Hz.

$w[n] \longleftrightarrow W(e^{j2\pi f\frac{T}{N}})$ – periodic in frequency with period $\frac{N}{T}$ Hz.

$z(t) \longleftrightarrow Z(2\pi f)$ (or $Z(f)$) – NOT periodic in frequency $Z(j\omega) \approx 0 \quad |\omega| > \frac{1}{2T}$

NB: remember time (frequency) $\longleftrightarrow$ frequency (time)

periodic $\longleftrightarrow$ discrete

Working backwards, mathematically

Analog Interpolation

$$z(t) = \sum_{k=-\infty}^{\infty} w[k]\, g\left(t - k\frac{T}{N}\right)$$

(assuming causality) (w[k]=0, k<0)

$$= \sum_{k=0}^{\infty} w[k]\, g\left(t - k\frac{T}{N}\right) = W\left(e^{j2\pi f\frac{T}{N}}\right) G(j2\pi f) = Z(f)$$

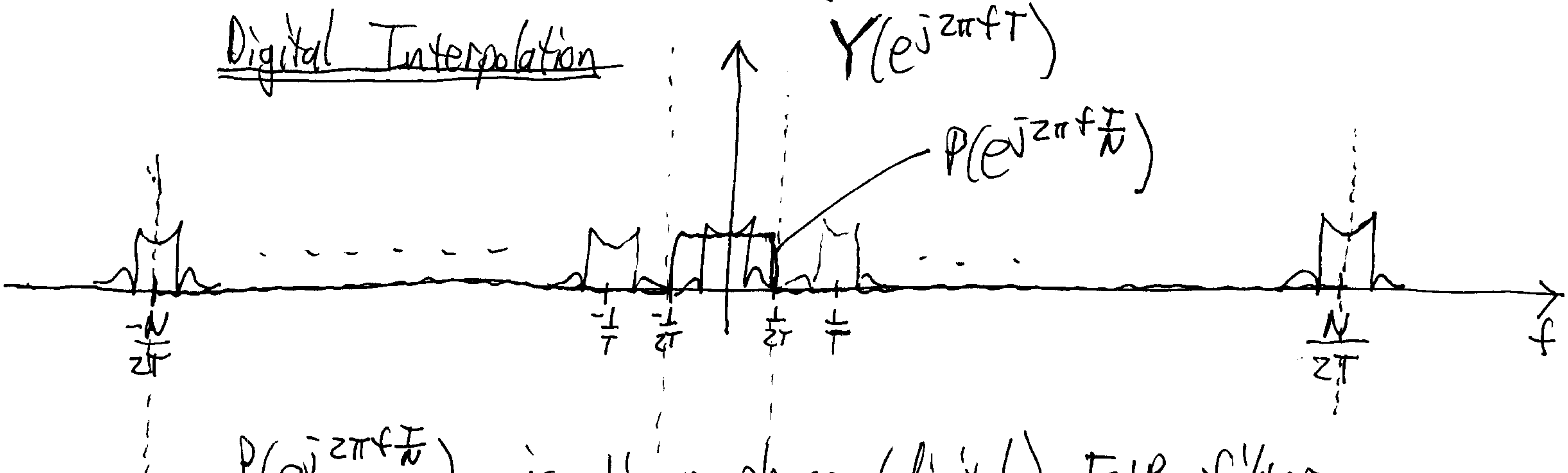
Digital Interpolation

$$w[n] = \sum_{k=0}^{\infty} y[k]\, p[n - kN] \longleftrightarrow Y\left(e^{j2\pi f\frac{T}{N}N}\right) P\left(e^{j2\pi f\frac{T}{N}}\right)$$

$$= W\left(e^{j2\pi f\frac{T}{N}}\right)$$

Working forwards pictorially in frequency

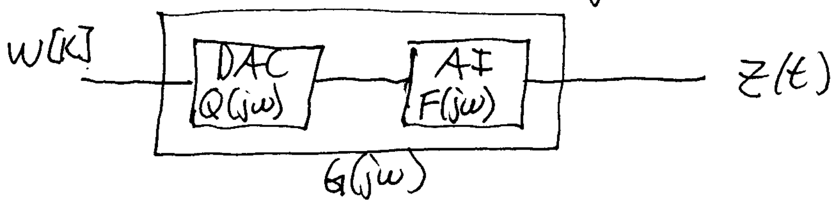
Digital Interpolation



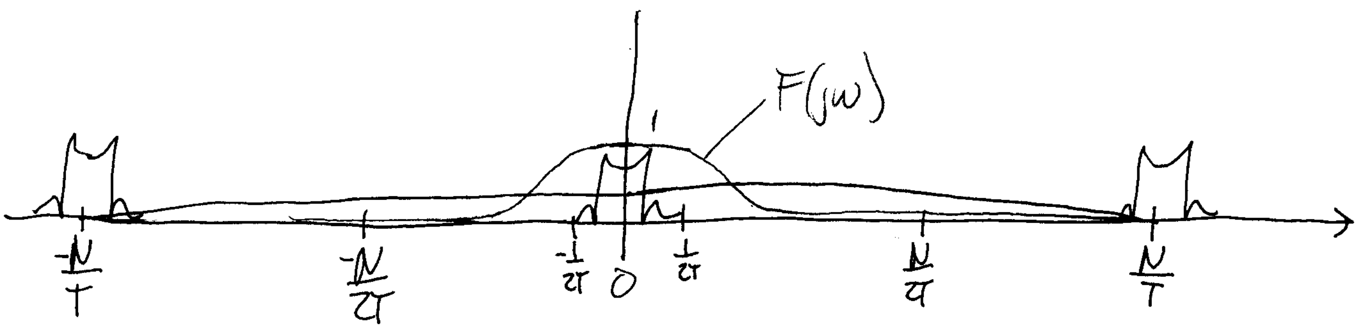
$P\left(e^{j2\pi f\frac{T}{N}}\right)$ is linear phase (digital) FIR filter, which means that there will be no "phase distortion" from this filter. (Later in Lab 4)

Analog Interpolation

Note $G(j\omega)$ is actually two filters



$$G(j\omega) = F(j\omega)Q(j\omega) \longleftrightarrow g(t) = (f * q)(t)$$

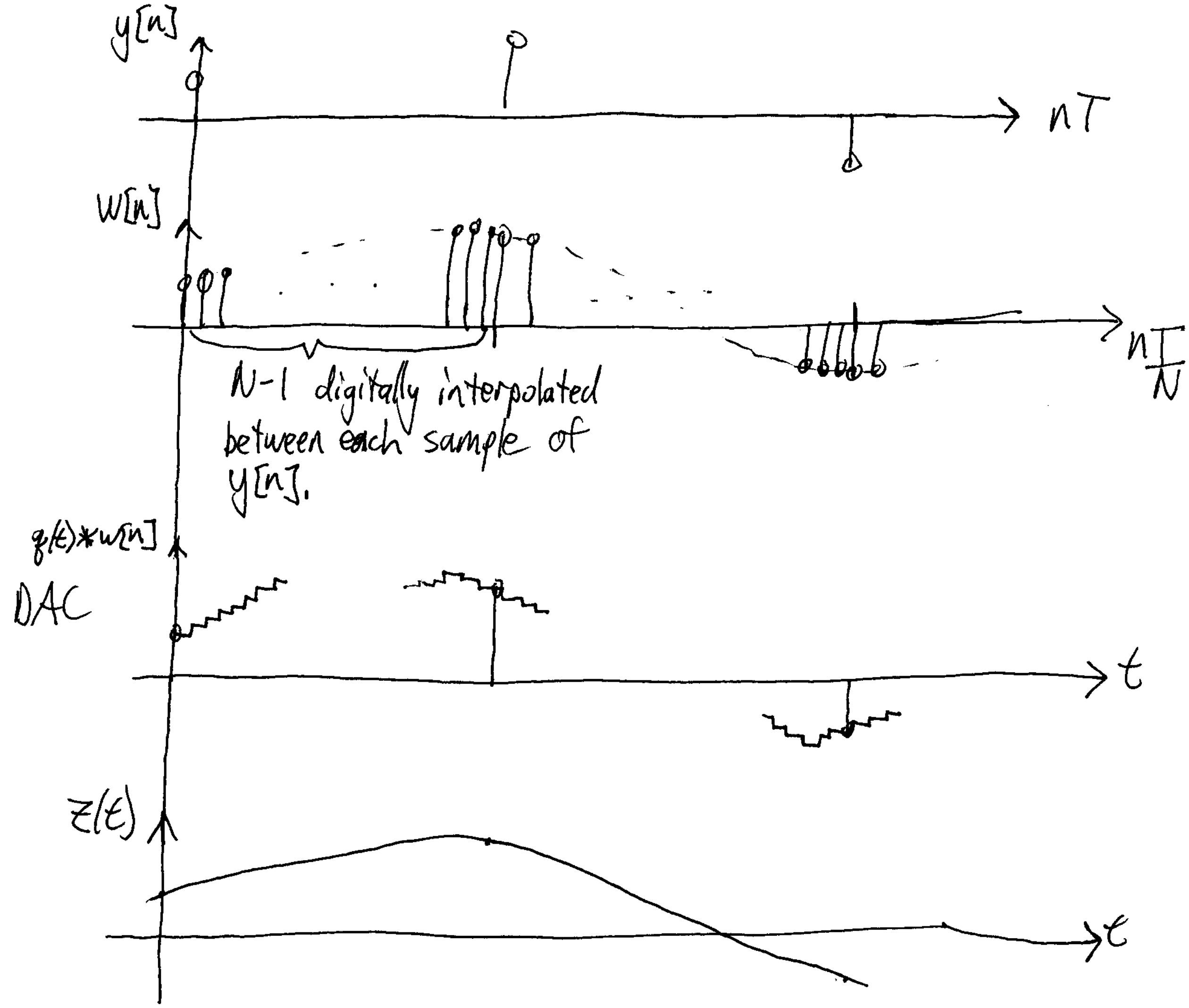


$$q(t) = \begin{cases} 1 & 0 \le t \le \frac{N}{T} \\ 0 & \text{o.w.} \end{cases} \longleftrightarrow Q(j\omega) = \frac{T}{N} e^{-j2\pi f \frac{T}{N}} \operatorname{sinc}\left(\pi f \frac{T}{N}\right)$$

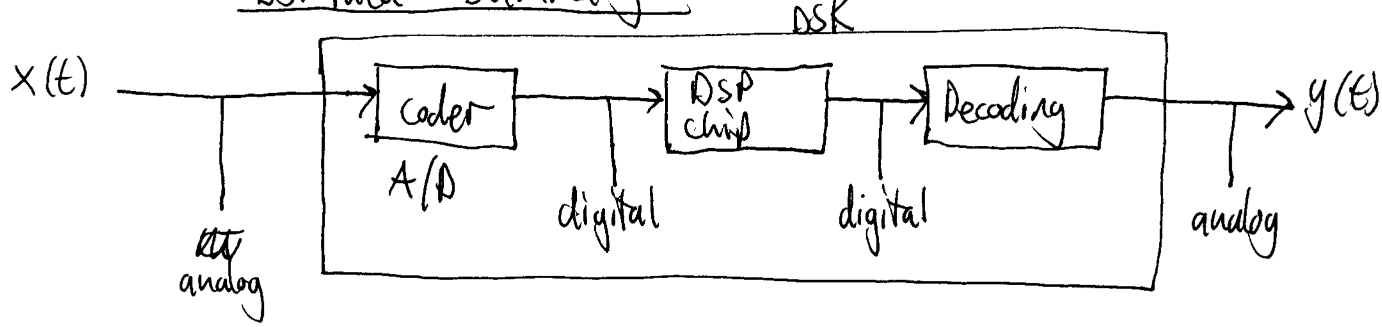
zero-order hold (ZOH)

$$F(j\omega) \approx 1, \quad |f| < \frac{1}{2T}$$

In the time domain



DSP idea - summary



Comments:

- A/D & D/A (coding & decoding) done by the codec (a single chip located on the DSK board)
- $y(t) \cong \hat{y}(t)$ (from before) when $x(t)$ is approximately bandlimited to the half sampling frequency.
  - In real applications, f_s would be a design parameter. However, in this class, f_s will be fixed.
- DSP chip implements the ~~algorithm~~ digital algorithm. This algorithm may be
  - LTI ($H(z)$), or
  - NLTV (non-linear time varying)
- In Lab 2, when I/O is performed, the DSP chip will be used as a straight wire (i.e. $H(z)=1$). For Labs 4-6, the DSP chip will be an LTI filter $H(z)$. In Lab 7, the DSP chip will be used to implement general NLTV ~~sy~~ communication systems