Fourier Transform of 'Shah' Sampling Function

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An infinite train of continuous-time Dirac delta functions ('Shah' or Comb function) is commonly used as a sampling function. Time-domain sampling of an analog signal produces artifacts which must be dealt with in order to faithfully represent the signal in the digital domain. These artifacts are most easily understood in the frequency domain, i.e. by looking at the Fourier transform of the Shah function and it's impact on the input signal. That is the purpose of this note (see Reference at end).

The Shah function in the time domain can be written as

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s), \tag{1}$$

where T_s is the sampling period.

Since $s(t)$ is periodic with a fundamental frequency $F_s = 1/T_s$, we can write the Fourier series expansion:

$$s(t) = \sum_{n=-\infty}^{\infty} s_n e^{j2\pi nF_s t}, \tag{2}$$

with the Fourier coefficients s_n given by

$$\begin{aligned} s_n &= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} s(t)e^{-j2\pi nF_s t}dt, \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-T_s/2}^{T_s/2} \delta(t - nT_s)e^{-j2\pi nF_s t}dt, \end{aligned} \tag{3}$$

and where Eqn(1) has been used to substitute for $s(t)$ and the integral has been brought inside the summation.

Now $\delta(t - T_s)$ will be non-zero over the interval $-T_s/2 \leq t < T_s/2$ only when $n = 0$ and $t = 0$. Therefore $s_n = 1/T_s$ for all n.

Therefore, the Fourier series expansion for the Shah function is

$$s(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{j2\pi nF_s t}. \tag{4}$$

For analyzing the effect of sampling and determining the required reconstruction filter, we need the representation of the Shah function in the frequency domain. This is given by the Fourier transform $S(j2\pi f)$, which is the Laplace transform evaluated on the imaginary axis of the s-plane.

$$
\begin{aligned}
S(j2\pi f) &= \int_{-T_s/2}^{T_s/2} s(t) e^{-j2\pi n F_s t} dt, \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-T_s/2}^{T_s/2} e^{-j2\pi(f+nF_s)t} dt.
\end{aligned} \tag{5}
$$

Now, the exponential function inside the integral can be written as the product of two exponentials. From the orthogonality of the complex exponentials,

$$
\int_{-T_s/2}^{T_s/2} e^{-j2\pi ft)} e^{nF_s t} dt = \delta(f + nF_s) \tag{6}
$$

and therefore

$$
S(j2\pi f) = F_s \sum_{n=-\infty}^{\infty} \delta(f + F_s). \tag{7}
$$

Reference:

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