Statistical Design of Nonrecursive Digital Filters

DAVID C. FARDEN, STUDENT MEMBER, IEEE, AND LOUIS L. SCHARF, MEMBER, IEEE

Abstract—**The problem of designing a finite duration impulse response (FIR) digital filter to approximate a desired spectral response is treated in this paper. The philosophy adopted is that for a given FIR filter structure, the filter coefficients can be designed to provide a minimum mean-squared error (MMSE) estimate of a random signal sequence (the design-signal) imbedded in a random noise sequence. By treating the signal and noise covariance functions as design parameters, one can design FIR filters with spectral responses that approximate the power spectral density of the design-signal. For signal processing applications that require some attention to signal fidelity, as well as noise rejection, the MMSE philosophy seems appropriate (as opposed to a maximum signal-to-noise ratio philosophy, for example).**

Several practical designs are presented that emphasize the simplicity of the design technique and illustrate the selection of design parameters. The designs show quite dramatically that the MMSE design technique can be competitive with existing low-pass and bandpass design techniques.

Finally, considerable attention is given to an efficient Toeplitz matrix inversion algorithm that permits rapid inversion of the covariance matrices that arise in the MMSE design. The resulting computation times for the design of high-order filters ($N = 128$, e.g.) appear to be shorter than computation times for competing algorithms.

I. INTRODUCTION

THE PROBLEM of designing digital filters to approximate "ideal" spectral characteristics has received considerable attention in recent years (e.g., [1],[2]). Many different approaches have been used in the design of finite duration impulse response (FIR) filters [3]–[13]. The majority of these FIR filter design techniques employ either an iterative frequency sampling technique or a window technique as, for example, in [6] and [11], respectively. An alternative approach adopted in this paper is to treat the FIR filter as an estimator structure to provide a minimum mean-squared error (MMSE) estimate of some design-signal. The MMSE philosophy has enjoyed great popularity in control and communication applications where signal fidelity is often an important consideration. More recently, the MMSE philosophy has found its way into the digital filtering literature [8]. The present paper is related in philosophy to [8], but extends the MMSE methodology to the design of linear phase MMSE low-pass and bandpass FIR filters with computation design times and spectral characteristics which appear to be competitive with existing design techniques. The large number of examples in [6] has prompted the choice of examples in this paper to coincide with those of [6] for comparison purposes. The choices of design parameters in this paper which lead to low-pass and bandpass designs incorporate the idea of a "don't care" region or a transition band which is related to the incompletely specified least mean square error method presented in [10].

Methodology

The underlying theme of this paper is the use of an hypothetical design-signal that has a power spectral density equal to the desired spectral response of the FIR filter. The design-signal is assumed to be a stationary random sequence imbedded in an additive stationary noise sequence and/or a strong adjacent-band interference sequence. The goal of the filter design, then, is to choose the filter weighting coefficients so that the filter output will be an MMSE estimate of the design-signal. By treating the noise and interference spectra as design parameters one can achieve a family of FIR filter designs and then choose the design that best satisfies the requirements of a particular application.

It should be emphasized that specifications such as in-band and out-of-band ripple, detailed transition-band behavior, and other more classical digital filter specifications do not (and, indeed, need not) arise in our procedures. However, by judiciously choosing signal, noise, and interference spectra, the designer can exercise implicit control over such specifications and obtain filters with excellent passband ripple and side-lobe characteristics.

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The authors are with the Department of Electrical Engineering, Colorado State University, Fort Collins, Colo. 80521.

Results

Several MMSE designs are presented for low-pass and bandpass FIR digital filters. One comb filter is designed to illustrate the versatility of the method. Each design assumes a perfectly bandlimited signal with known center frequency and bandwidth. Similar specifications are often employed in more conventional designs.

The results presented in this paper indicate that clever choices of signal, noise, and interference spectra can lead to designs that are competitive with other published designs. A particularly attractive feature of the results is the modest computational times required to realize the filters of interest. Each design requires a simple matrix inversion that is effected very efficiently with a recently published Toeplitz matrix inversion algorithm [14].

II. THE FILTER MODEL

An FIR tapped delay line filter structure is illustrated in Fig. 1. The structure consists of a tapped delay line with N taps, $N - 1$ ideal time delays (D), and N weights $\{w_i, i = 1,2,\cdots,N\}$. The input to the filter is assumed to be a stationary random sequence, $\{x(kT), k = 0, \pm 1, \pm 2,\cdots\}$, with sampling interval T. Note that D must be an integer multiple of T. It will be convenient to denote the N filter weighting coefficients by the $(N \times 1)$ vector,

$$W = (w_1,w_2,\cdots,w_N)', \tag{1}$$

where $'$ denotes transpose.

The data at the N taps of the tapped delay line at time kT can be represented by the $(N \times 1)$ vector,

$$X(kT) = S(kT) + N(kT) \tag{2}$$

where $S(kT)$ and $N(kT)$ are, respectively, signal and noise components of the data vector:

$$\begin{aligned} S'(kT) &= [s_1(kT), s_2(kT),\cdots,s_N(kT)] \\ N'(kT) &= [n_1(kT), n_2(kT),\cdots,n_N(kT)]. \end{aligned} \tag{3}$$

Signal and Noise Covariance Functions

For convenience, the signal and noise sequences $\{s_i(kT),k = 0, \pm 1,\pm 2,\cdots\}$ and $\{n_i(kT), k = 0,\pm 1, \pm 2,\cdots\}$ are assumed to be samples of uncorrelated, zero-mean, wide-sense stationary continuous time random processes, $s(t)$, and $n(t)$, respectively. Define the signal and noise covariance functions, $R_s(\tau)$ and $R_n(\tau)$ by

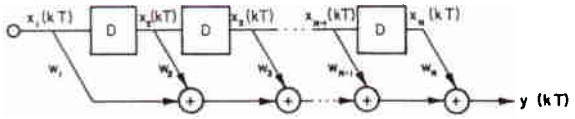
$$\begin{aligned} R_s(\tau) &= E\{s(t)s(t + \tau)\}, \\ R_n(\tau) &= E\{n(t)n(t + \tau)\}, \end{aligned} \tag{4}$$

where $E\{\cdot\}$ denotes statistical expectation.

Define the data covariance matrix R_{XX} by

$$\begin{aligned} R_{XX} &= E\{X(kT)X'(kT)\} \\ &= R_{SS} + R_{NN}, \end{aligned} \tag{5}$$

where the signal and noise covariance matrices are, respectively,



N = Number of taps on delay line
D = Time delay between adjacent taps
T = Sampling interval

Fig. 1. Filter model.

$$\begin{aligned} R_{SS} &= E\{S(kT)S'(kT)\} \\ R_{NN} &= E\{N(kT)N'(kT)\}. \end{aligned} \tag{6}$$

It is noted that $trR_{SS} = N\sigma_s^2$ and $trR_{NN} = N\sigma_n^2$, where σ_s^2 is the signal variance per tap and σ_n^2 is the noise variance per tap. The ijth elements of R_{SS} and R_{NN} are given by $R_s[(i - j)D]$ and $R_n[(i - j)D]$, respectively.

One other statistical parameter of interest is the so-called signal covariance vector

$$P_{sl} = E\{s_l(kT)S(kT)\}. \tag{7}$$

Noting that the ith element of P_{sl} is given by $R_s[(i - l)D]$, the interpretation of $s_l(kT)$ can be generalized to include noninteger l, corresponding to $s_1(kT - (l - 1)D)$.

Filter Frequency Response

The magnitude-squared frequency response, $|H(f)|^2$, of the FIR filter is defined to be

$$|H(f)|^2 = \left|\sum_{k=1}^{N} w_k \exp\{-j2\pi f(k - 1)D\}\right|^2. \tag{8}$$

It is worth emphasizing that for $f = l/ND$, $l = 0,1,\cdots$, $N - 1$, $H(f)$ is the discrete Fourier transform (DFT) of W. A computationally efficient method of computing (8) for a large number of equally spaced frequency points is to augment W with zeros, i.e., replace W' by $(w_1,w_2,\cdots,w_N,0,0,\cdots,0)$ and use the fast Fourier transform (FFT).

III. THE MMSE FILTER DESIGN TECHNIQUE

The MMSE problem is one of designing the weight vector W so that the filter output $y(kT)$ (see Fig. 1) is the MMSE estimate of the design-signal sequence $\{s_l(kT), k = 0, \pm 1, \pm 2, \cdots\}$. Recall that $s_l(kT)$ is the design-signal observed at time $kT - (l - 1)D$ at the input of the tapped delay line.

The output of the filter, $y(kT)$, is given by

$$y(kT) = W'X(kT). \tag{9}$$

Therefore, the mean-squared error in estimating $s_l(kT)$ with $y(kT)$ is

$$\begin{aligned} \xi(W) &= E\{[y(kT) - s_l(kT)]^2\} \\ &= W'R_{XX}W - 2W'P_{sl} + \sigma_s^2, \end{aligned} \tag{10}$$

where $\sigma_s^2 = R_s(0)$. Taking the gradient of $\xi(W)$ with respect to W, one can easily show that the well known

Wiener solution for the minimizing weight vector is

$$W_{MMSE} = R_{XX}^{-1}P_{sl}. \tag{11}$$

Computational Considerations

The design of an MMSE filter requires inversion of the $(N \times N)$ covariance matrix R_{XX}. Recently, a computationally efficient algorithm [14] has been développed for the inversion of such matrices. The inversion algorithm exploits the Toeplitz nature of R_{XX}, i.e., the algorithm makes efficient use of the fact that r_{ij}, the ijth element of R_{XX} depends only on $|\, i - j \,|$:

$$\begin{aligned} r_{ij} &= E\{x_i(kT)x_j(kT)\} \\ &= E\{x_i(kT)x_i[kT + (j - i)D]\} \\ &= R_x(|\, i - j \,|\, D) \end{aligned} \tag{12}$$

where $R_x(\cdot)$ is the data covariance function, i.e., $R_x(\cdot) = R_s(\cdot) + R_n(\cdot)$. A Fortran subroutine making use of the algorithm is presented in the Appendix. The routine is capable of computing the solution of (11) for $N = 256$ in less than four s of central processor time on a CDC 6400 computer.

A Sufficient Condition for Linear Phase

A well known result for FIR filters is that a sufficient condition for linear phase is that the weight vector be symmetric about its center, i.e.,

$$w_i = w_{N-i+1} \tag{13}$$

for all $i \in \{1,2,\cdots,N\}$. It has been shown [15] that the inverse of a persymmetric (symmetric about both main diagonals) matrix is also persymmetric. Since R_{XX} is persymmetric it follows easily that if the elements of P_{sl} satisfy (13), then a filter designed using (11) will have linear phase. For N even or odd, choosing the ith element of P_{sl} to be $R_s[(i - (N + 1)/2)D]$ will result in linear phase MMSE filters.

IV. DESIGN CONSIDERATIONS

As the discussion of the previous sections has shown, there are several design parameters that must be rationally selected in the design of an MMSE filter. In this section we discuss these design parameters and show how one can choose $R_s(\tau)$ and $R_n(\tau)$ to achieve effective low-pass and bandpass designs.

Signal Covariance Function, $R_s(\tau)$

The signal covariance function $R_s(\tau)$ is uniquely determined as the inverse Fourier transform of the desired magnitude-squared frequency function of the FIR filter between $-f_f$ and f_f, where $f_f = 1/2D$ is the foldover frequency of the FIR filter. For example, consider the perfectly bandlimited spectrum $S(f)$ illustrated with the solid lines of Fig. 2(a). The inverse Fourier transform of $S(f)$, $R_s(\tau)$, is given by

$$R_s(\tau) = 2BW \frac{\sin(\pi BW\tau)}{\pi BW\tau} \cos 2\pi f_0\tau. \tag{14}$$

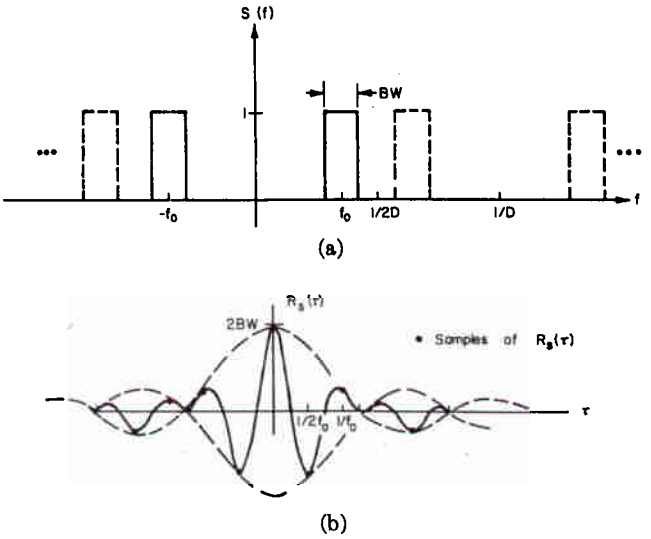


Fig. 2. Signal spectral model for bandpass designs. (a) Desired spectral response, $S(f)$. (b) Inverse transform of $S(f)$.

The function $R_s(\tau)$ illustrated in Fig. 2(b) may be thought of as the covariance function of some zero-mean, wide-sense stationary design-signal $s(t)$ with power spectral density $S(f)$. The spectral region where $S(f)$ is nonzero will be referred to as the passband. It is obvious that such a selection of $S(f)$ will lead to band-pass filter designs.

Noise Covariance Function, $R_n(\tau)$

Once $R_s(\tau)$ is fixed, the noise covariance function $R_n(\tau)$ may be chosen to reflect the relative costs of passing a component of the noise and distorting the signal. For conventional low-pass and bandpass designs, $R_n(\tau)$ will be the inverse Fourier transform of the power spectral density, $S_n(f)$ of a broadband interference. $S_n(f)$ will be nonzero in a spectral region which will be referred to as the stopband, as illustrated in Fig. 3. The spectral level of $S_n(f)$ will determine the price paid for passing a signal in the stopband relative to the price paid for distorting a signal in the passband; thus the level of $S_n(f)$ is a free design parameter. The spectral region between the passband and the stopband will be referred to as the transition band. These heuristic comments are formalized in the following paragraph.

From (11), the mean-squared error in estimating $s_l(kT)$ when only the signal is present is given by

$$\begin{aligned} \xi_s(W_{MMSE}) &= E\{[W_{MMSE}'S(kT) - s_l(kT)]^2\} \\ &= W_{MMSE}'R_{SS}W_{MMSE} - 2W_{MMSE}'P_{sl} + \sigma_s^2. \end{aligned} \tag{15}$$

From the definitions of R_{SS} and P_{sl} it should be clear that one may interpret $\xi_s(W_{MMSE})$ as "passband distortion." Making use of the well known fact that the mean-squared error, $\xi(W_{MMSE})$ is always less than or equal to the signal variance, σ_s^2, we have

$$0 \leq \xi_s(W_{MMSE}) + W_{MMSE}'R_{NN}W_{MMSE} \leq \sigma_s^2 \tag{16}$$

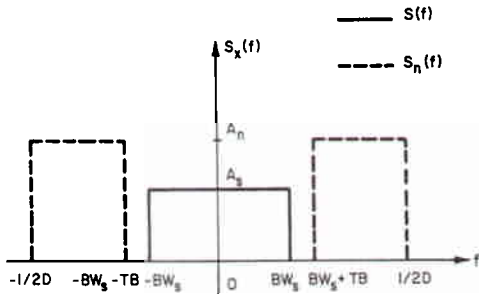


Fig. 3. Spectral model for low-pass filter designs.

where $W_{MMSE}'R_{NN}W_{MMSE}$ is the error variance due to the stopband noise (or broadband interference). If $R_{NN} = A_n\rho_{NN}$ is the noise covariance matrix for a broadband interference of spectral height A_n, (16) becomes

$$0 \leq W_{MMSE}'\rho_{NN}W_{MMSE} \leq \frac{1}{A_n}[\sigma_s^2 - \xi_s(W_{MMSE})] \leq \frac{\sigma_s^2}{A_n} \tag{17}$$

and it follows that increasing A_n will decrease $W_{MMSE}'\rho_{NN}W_{MMSE}$, which is directly related to the stopband sidelobe level, at the expense of increasing $\xi_s(W_{MMSE})$, the passband distortion.

Estimation Point, l

Filter applications for which a delayed estimate of the signal component at the input of the FIR filter is acceptable permit a choice of delay $(l-1)D$ that can further reduce the mean-square estimation error and, hence, yield a more desirable spectral response than other choices of delay. From (11) it is easily established that the minimum mean-square error, $\xi(W_{MMSE})$, in estimating $s_l(kT)$ is given by

$$\begin{aligned}\xi(W_{MMSE}) &= \sigma_s^2 - W_{MMSE}'P_{sl} \\ &= \sigma_s^2 - P_{sl}'R_{XX}^{-1}P_{sl}.\end{aligned} \tag{18}$$

The choice of l that minimizes $\xi(W_{MMSE})$ is that value of l for which the quadratic form $P_{sl}'R_{XX}^{-1}P_{sl}$ is maximized. Rather than maximize $P_{sl}'R_{XX}^{-1}P_{sl}$, a procedure that depends on R_{XX}^{-1}, consider maximizing the following bound:

$$\begin{aligned}P_{sl}'R_{XX}^{-1}P_{sl} &= tr\,\{R_{XX}^{-1}P_{sl}P_{sl}'\} \\ &\leq tr\,\{R_{XX}^{-1}\}tr\{P_{sl}P_{sl}'\} \\ &= tr\,\{R_{XX}^{-1}\}P_{sl}'P_{sl}.\end{aligned} \tag{19}$$

From (19), a logical choice for the delay $(l-1)D$ is one that maximizes the norm of P_{sl}. Choosing l as near as possible to the center of the tapped delay line results in an excellent approximation to the maximum achievable $\|P_{sl}\|^2$. This choice coincides nicely with the discussion in Section III which suggested that for $l = (N+1)/2$, linear phase filters would result.

Normalization Procedures

The level of the MMSE filter spectral response (equivalently, the norm of the weight vector) depends on the signal and noise covariance matrices. Consequently, the weight vector designed by the MMSE method must be scaled in order to achieve the desired spectral level. One approach to scaling would be to select a frequency, f_d, where a spectral level, A, is required, and then compute a scaled weight vector

$$W_s = W\left(\frac{A}{S(f_d)}\right)^{1/2} \tag{20}$$

where $S(f_d)$ is the spectral response of W at $f = f_d[S(f_d) = |H(f_d)|^2$ from (8)].

An alternative approach is to require that the output variance of the filter be σ_s^2 when driven by a discrete white noise process $n(kT)$ of unit spectral height. It follows easily that the noise covariance matrix R_{NN} of (6) is given by $R_{NN} = (2/D)I$. Thus the filter output variance is given by

$$\begin{aligned}\sigma_y^2 &= W'R_{NN}W \\ &= \frac{2}{D}W'W.\end{aligned} \tag{21}$$

Now consider use of the following scaled weight vector:

$$W_s = W\left(\frac{D\sigma_s^2}{2W'W}\right)^{1/2}. \tag{22}$$

It follows from (22) that the resulting filter output variance is $\sigma_y^2 = \sigma_s^2$.

V. DESIGN PERFORMANCE

In this section we discuss the performance of the MMSE-designed low-pass filter for several choices of design parameters. The spectral model of Fig. 3 is used throughout this section. A flow chart for the MMSE design technique is shown in Fig. 4. More conventional design techniques begin the design with specification of filter bandwidth, transition band, and allowable passband ripple and sidelobe height; the present technique calls for specification of the intended signal and noise environment in which the filter is to be used. By treating the noise environment as a design parameter, one can obtain a family of filter designs and observe passband ripple and maximum sidelobe level for comparison with more conventional designs.

Performance Measures

A typical magnitude-squared frequency response $|H(f)|^2$ is shown as the solid line in Fig. 5. The design-signal power spectral density, $S(f)$, is shown as the dotted line. We define the passband ripple, R_1, as

$$R_1 = 10\log_{10}\left\{\underset{f_1,f_2\in[0,BW_s]}{\text{maximum}}\frac{|H(f_1)|^2}{|H(f_2)|^2}\right\}. \tag{23}$$

The maximum sidelobe height, S_1, is defined by

$$S_1 = 10\log_{10}\{\underset{f\in[BW_s+TB,f_l]}{\text{maximum}}|H(f)|^2\}. \tag{24}$$

Another performance measure of interest is the maximum

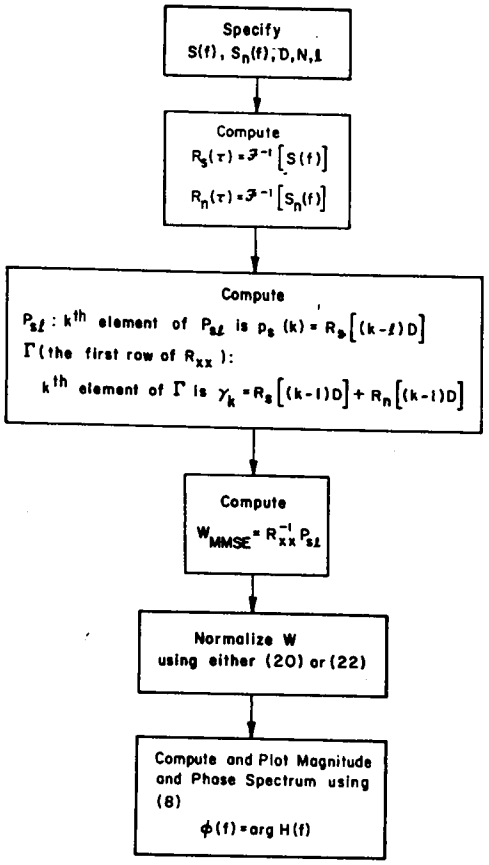


Fig. 4. Flow chart for the MMSE design technique.

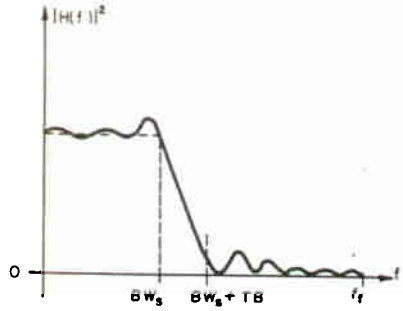


Fig. 5. Typical magnitude-squared transfer function for FIR filter.

of $|H(f)|^2$ for frequencies greater than the first frequency, f_s, greater than BW_s at which the slope of $|H(f)|^2$ changes from negative to positive. Define S_2 by

$$S_2 = 10 \log_{10} \{\text{maximum}_{f \in [f_s, f_f]} |H(f)|^2\}. \quad (25)$$

The performance measures S_2 and f_s are of interest because the *effective* transition band (defined as $f_s - BW_s$) cannot be specified *a priori*.[1] However, in all cases tried the difference between $f_s - BW_s$ and TB was less than $1/2N$. We reiterate that this discrepancy between TB and $f_s - BW_s$ in no way detracts from the optimality of the filter, i.e., the filter provides an MMSE estimate of the low-pass process characterized by the signal spectrum of Fig. 3 when the process is immersed in bandpass, adjacent channel interference.

[1] Actually the discrepancy between effective transition band and design transition band is not unique to this design. That is, other design procedures often lead to slight differences between TB and $f_s - BW_s$; in fact, there appears to be no universal agreement concerning the proper definition of effective transition band.

Effects of Parameters on the Performance Measures

A large number of low-pass filters have been designed to illustrate the effects of the design parameters on the passband ripple and sidelobe level. Specifically, filters have been designed for $BW_s \in \{0.1,0.2,0.3,0.4,0.5\}$, $f_f = 1.0$, $A_s = 1.0$, $A_n \in \{10^3,10^6,10^9\}$, $N \in \{32,64,128,256\}$ and $TB \in \{2/N,4/N,6/N,8/N,10/N\}$. Note that a different set of transition bands has been used for each value of N. The results indicate no dramatic difference in maximum sidelobe height or passband ripple for the values of bandwidth used.[2] Fig. 6 shows the effects of the parameters A_n, the interference level, and TB, the transition band, on the passband ripple, R_1, for $N \in \{32,64,128,256\}$, and $BW_s \in \{0.1,0.2,0.3,0.4,0.5\}$. Fig. 7 shows the effects of the parameters A_n and TB on the maximum sidelobe level, S_1. Similarly, Fig. 8 shows the effects of the parameters A_n and TB on the performance measure S_2. Also shown in Figs. 7 and 8 is the range of S_1 from the data given in [6] for $BW_s \in \{0.1,0.5\}$, $TB \in \{4/N,6/N,8/N\}$, and $N \in \{32,64,128,256\}$. Figs. 7 and 8 illustrate the competitive nature of the MMSE filter when rated according to conventional filter performance measures.

Example Designs

In order to illustrate the performance of the MMSE design technique, several practical designs are shown in Figs. 9–11.

Figs. 9(a) and 9(b) show $|H(f)|^2$ for a low-pass filter designed with $BW_s = 0.2$, $TB = 1/16$, and $A_n = 10^3$ for $N = 32$ and 128 respectively; these designs illustrate the effect of increasing N without changing any other parameter. Fig. 9(c) shows a low-pass filter designed with $BW_s = 0.2$, $TB = 1/32$, and $A_n = 10^3$ for $N = 256$. Comparing Figs. 9(b) and 9(c), we note that decreasing TB by a factor of 2 and increasing N by a factor of 2 does not markedly change any performance measure, except the effective transition band, as one would expect from Figs. 6–8.

Fig. 10(a) shows the appropriate spectral model for bandpass designs. Figs. 10(b) and 10(c) show $|H(f)|^2$ for $f_0 = 61/128$, $TB = 1/16$, $BW_s = 15/128$, $N = 128$, and $A_n = 10^6$ and 10^9 respectively. The above parameters were chosen to coincide with the designs given as Figs. 26–28 in [6]. Note the decrease in maximum sidelobe height and the increase in passband ripple with an increase in A_n.

A constant-Q comb filter was designed using the spectral

[2] The single exception to this is the case where $BW_s = 0.1$, $N = 32$, $TB = 1/16$, for which an increase in A_n beyond 10^3 makes no significant difference. For this reason this case is *not included* in any of the figures.

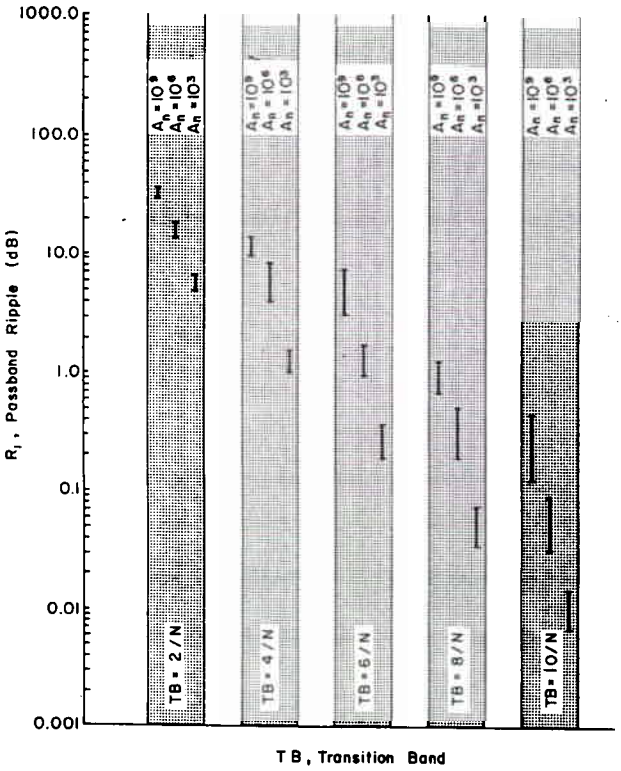


Fig 6. Passband ripple, R_1, as a function of transition band.

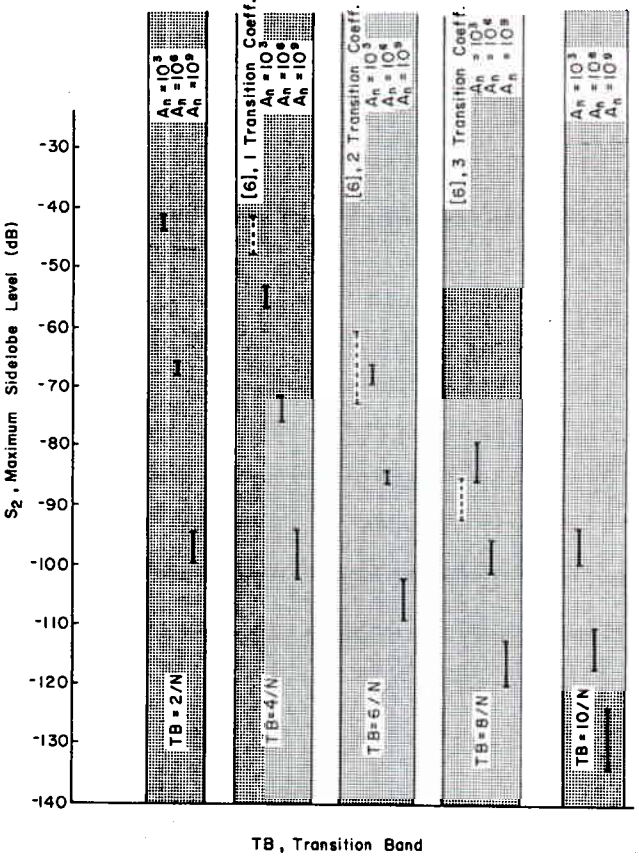


Fig. 8. Maximum sidelobe height, S_2, as a function of transition band.

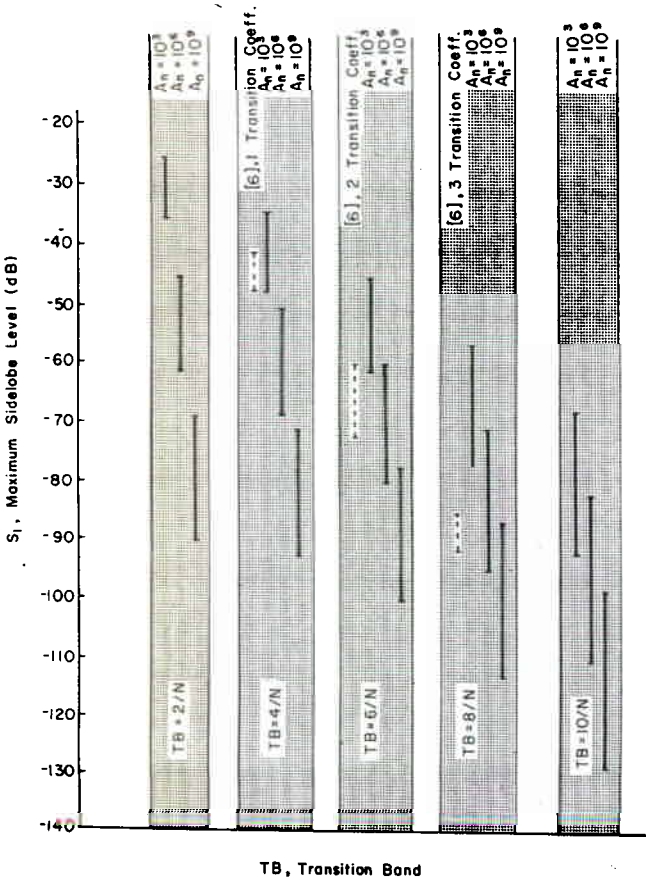


Fig. 7. Maximum sidelobe height, S_1, as a function of transition band.

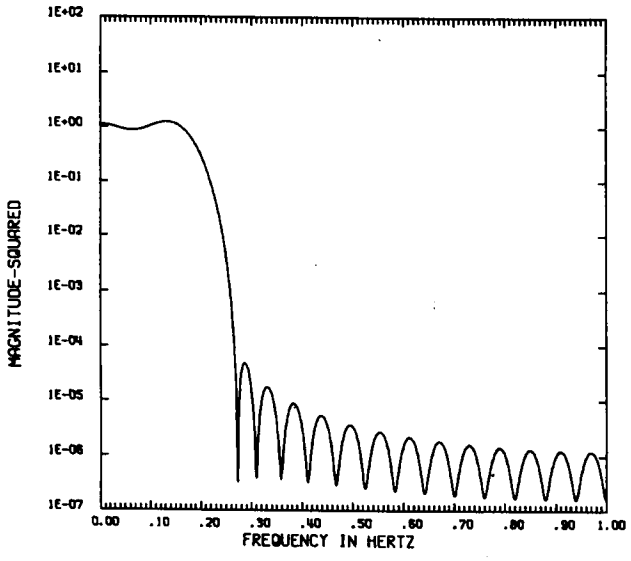
model of Fig. 11(a) with $A_n = 10^3$; the magnitude-squared frequency response of the MMSE filter is shown in Fig. 11(b). This example shows the versatility of the MMSE design technique for piecewise-constant spectral models. There is, however, no inherent reason why the design technique cannot be extended to include more general spectral models.

VII. CONCLUSIONS

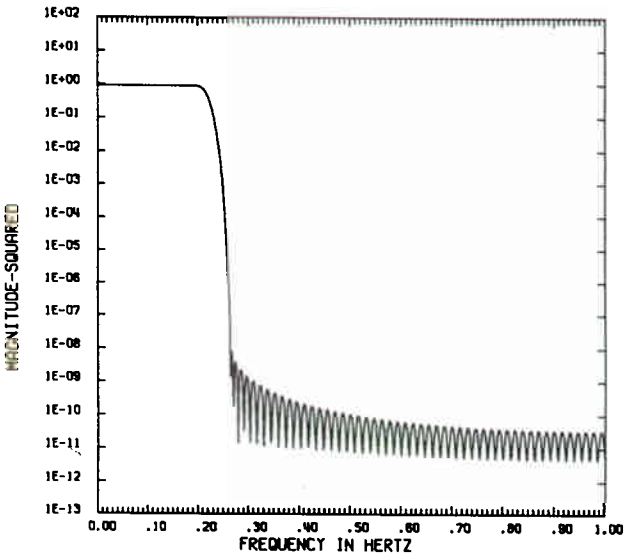
The design philosophy presented in this paper encourages the designer to treat the FIR filter as an estimator structure and to specify the free design parameters in a way directly related to the intended application of the filter. Some connections between these parameters and classical filter performance measures have been advanced.

Several designs have been presented for hypothetical signal-interference-noise applications where the motivation is to obtain effective MMSE estimates of low-pass and bandpass signals immersed in adjacent channel interference. By treating the interference level and bandwidth as design parameters, the authors have obtained low-pass bandpass, and comb filter designs with passband and stopband properties that are competitive with existing designs [6]. These properties are summarized in Figs. 6–11.

The computation times required to design the class of MMSE filters discussed in this paper appear to be markedly shorter than the times required for other FIR



(a)



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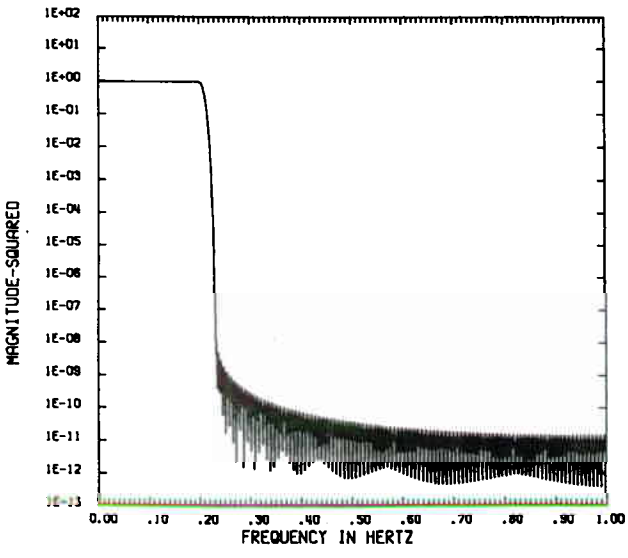
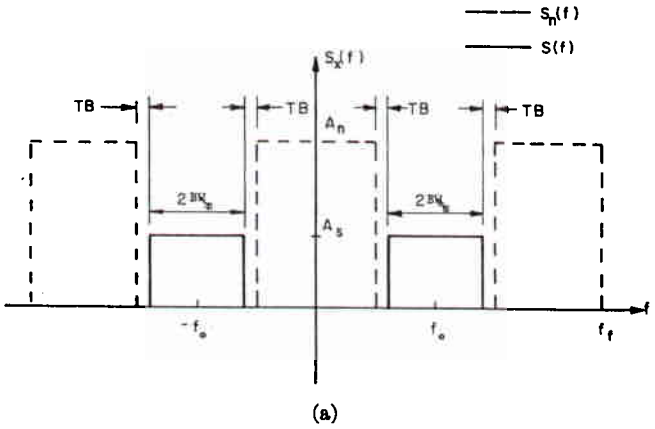
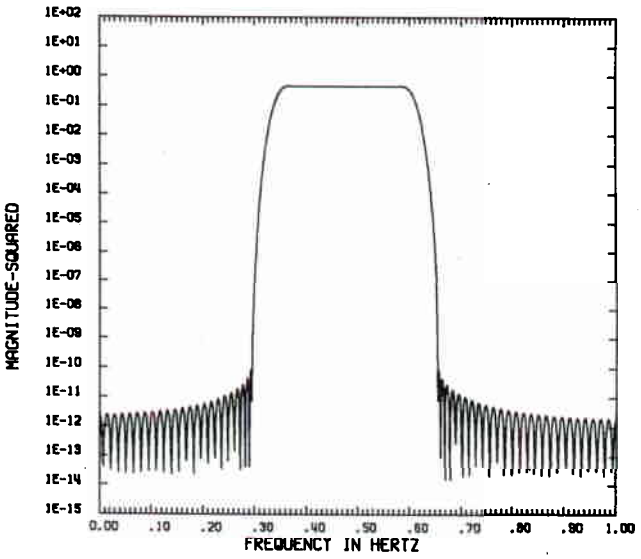


(c)

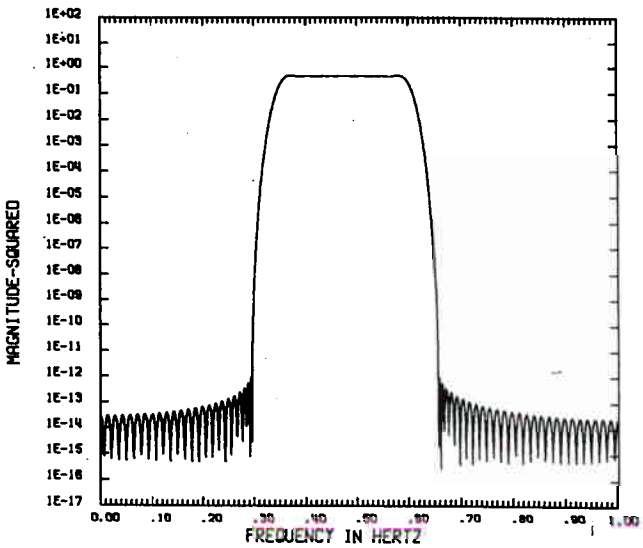
Fig. 9. Low-pass example designs: (a) $BW_s = 0.2$, $TB = 1/16$, $A_n = 10^3$, $N = 32$. (b) $BW_s = 0.2$, $TB = 1/16$, $A_n = 10^3$, $N = 128$. (c) $BW_s = 0.2$, $TB = 1/32$, $A_n = 10^3$, $N = 256$.



(a)



(b)



(c)

Fig. 10. (a) Spectral model for bandpass filter designs. (b) Bandpass example design: $f_0 = 61/128$, $TB = 1/16$, $BW_s = 15/128$, $N = 128$, $A_n = 10^6$. (c) Bandpass example design: $f_0 = 61/128$, $TB = 1/16$, $BW_s = 15/128$, $N = 128$, $An = 10^9$.

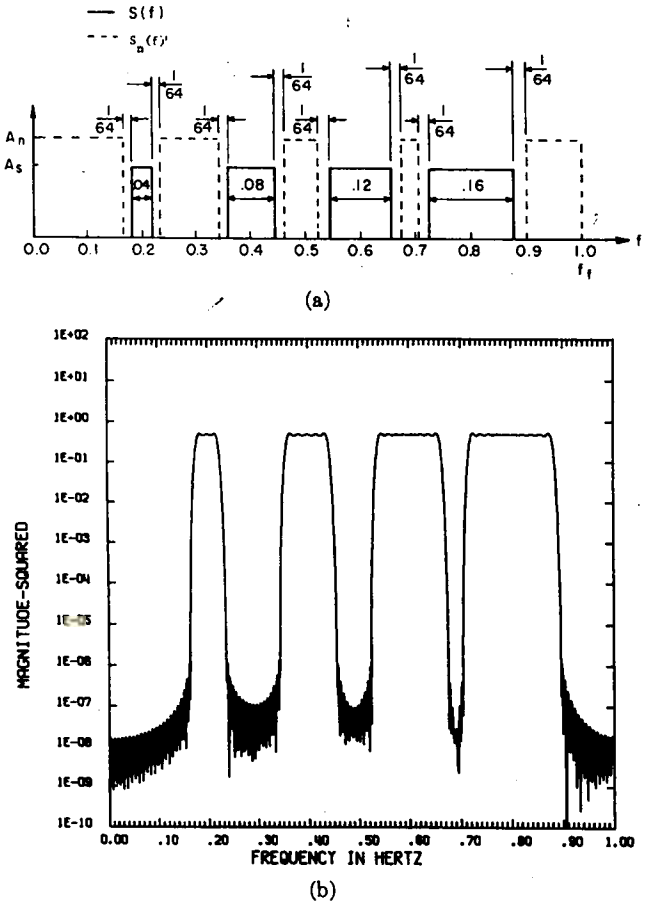


Fig. 11. (a) Spectral model for constant-Q comb filter. (b) Example design for constant-Q comb filter: $A_n = 10^3$, $N = 256$.

filter design techniques. Furthermore, the design times (less than four s for $N = 256$) are independent of the specified transition bandwidth, a property not shared by the design technique of [6], for example. The very short computation times for the MMSE filter designs accrue from special properties of the Toeplitz matrices that arise and appear to be a strong virtue of the MMSE designs for stringent filtering requirements which force the use of a high order filter.

On the pessimistic side, although the computation times for the design of large order filters seem to be extremely reasonable, the storage requirements (for the inverse matrix) increase as $[(N + 1)^2/4]$, where $[x]$ denotes "largest integer contained in x." For example, storage of the inverse matrix for $N = 256$ requires 16 512 words of storage, limiting the design technique to at least a medium scale computing facility. However, it seems reasonable to conclude that the inversion algorithm could be modified to reduce storage requirements (at the expense of increased computational time).

Finally, making use of the persymmetric nature of the covariance matrix that arises in the MMSE design procedure, the authors have established a sufficient condition for the design of linear phase MMSE filters and shown that such filters are easily obtained at no apparent expense to the overall design. In fact the motivation to seek linear phase designs is consistent with the quest for small-squared estimation errors.

APPENDIX

FORTRAN ROUTINE FOR TOEPLITZ MATRIX INVERSION

```
      SUBROUTINE TPLTZ (N,*,PS,A,*,*,MM)                                TPZ  10
C                                                                       TPZ  20
C     THIS SUBROUTINE COMPUTES THE NX1 VECTOR W DETERMINED BY           TPZ  30
C                                                                       TPZ  40
C                        PAP**-1*PS,                                    TPZ  50
C                                                                       TPZ  60
C     WHERE WXX IS AN NXN SYMMETRIC TOEPLITZ MATRIX WHOSE FIRST         TPZ  70
C     ROW IS R.  THE INVERSE OF WXX IS STORED IN B WHICH IS OF          TPZ  80
C     LENGTH MM=((N+1)**2)/4.  N MUST BE GREATER THAN OR EQUAL          TPZ  90
C     TO THREE.  A IS A WORK ARRAY OF DIMENSION N.                      TPZ 100
C                                                                       TPZ 110
C     THIS ROUTINE IS BASED ON AN ALGORITHM PRESENTED BY D. M. PREIS.   TPZ 120
C     IEEE TRANS. ANTENNAS PROPAGAT., MARCH 1972, PP. 204-206.          TPZ 130
C                                                                       TPZ 140
C     THIS ROUTINE WAS WRITTEN FOR A CDC 6400 AT COLORADO STATE         TPZ 150
C     UNIVERSITY BY D. C. FARDEN.                                       TPZ 160
C                                                                       TPZ 170
      DIMENSION R(N), PS(N), A(N), W(N), B(MM)                          TPZ 180
      IM1=N-1                                                           TPZ 190
      MM2=N-2                                                           TPZ 200
      C=R(1)                                                            TPZ 210
      CINV=1.0/C                                                        TPZ 220
      DO 101 I=1,IM1                                                    TPZ 230
  101 W(I)=CINV*R(I+1)                                                  TPZ 240
      AM=1.0-W(1)**2                                                    TPZ 250
      A(1)=-W(1)                                                        TPZ 260
      DO 105 M=1,MM2                                                    TPZ 270
         C1=0.0                                                         TPZ 280
         I1=M+1                                                         TPZ 290
         DO 102 L=1,M                                                   TPZ 300
  102    C1=C1+A(L)*R(I1-L)                                             TPZ 310
         AL=-(R(I1)+C1)/AM                                              TPZ 320
         C2=AL*CINV                                                     TPZ 330
         DO 103 L=1,M                                                   TPZ 340
  103    W(L)=A(L)+C2*A(I1-L)                                           TPZ 350
         DO 104 L=1,M                                                   TPZ 360
  104    A(L)=W(L)*CINV*(-1)                                            TPZ 370
         A(I1)=CINV                                                     TPZ 380
  105 AM=AM-(AL**2)*CINV                                                TPZ 390
      DO 106 I=1,N                                                      TPZ 400
  106 B(I)=A(N-I)                                                       TPZ 410
      B(1)=1.0/AM                                                       TPZ 420
      CONST=C*C                                                         TPZ 430
      DO 107 I=2,N                                                      TPZ 440
  107 B(I)=CINV*A(I-1)                                                  TPZ 450
      MM=(N+1)**2/4                                                     TPZ 460
      M1=M+1                                                            TPZ 470
      DO 108 L=2,M                                                      TPZ 480
         JM1=L-1                                                        TPZ 490
         JMAX=N-L                                                       TPZ 500
         ML=JMIN0(JMAX+1,M+1)                                           TPZ 510
         ML2=(L-1)*(JMAX+2)                                             TPZ 520
         DO 108 J=M1,ML                                                 TPZ 530
  108 B(J+ML2)=B(J+ML2)+CONST*(A(JM1+J)*A(JM1+L)-W(JM1+N)*W(JM1+N))     TPZ 540
      A(1)=2.0                                                          TPZ 550
      W(1)=0.0                                                          TPZ 560
      DO 109 J=1,N                                                      TPZ 570
         A(J+1)=W(J)                                                    TPZ 580
  109 W(J)=A(J)*PS(J)*A(J)                                              TPZ 590
      W(N)=W(N)+PS(N)*A(1)                                              TPZ 600
      DO 112 L=2,N/2+1                                                  TPZ 610
         L1=L-1                                                         TPZ 620
         LM=L-1                                                         TPZ 630
         MM=L                                                           TPZ 640
         ML=N-L+1                                                       TPZ 650
         W(L)=0.0                                                       TPZ 660
         A(L+1)=0.0                                                     TPZ 670
         DO 110 J=1,L                                                   TPZ 680
            JM1=J-1                                                     TPZ 690
            JS=JM1*(N-JM1)                                              TPZ 700
            JA=JS+L                                                     TPZ 710
            JA1=JS+LM                                                   TPZ 720
            W(J)=W(J)+PS(J)*A(JA)*PS(K)*A(JA)                           TPZ 730
  110    W(ML)=W(ML)+PS(K)*A(JA)*PS(J)*A(JA)                           TPZ 740
         JA=JA+ML-L                                                     TPZ 750
         DO 111 J=L,M                                                   TPZ 760
            W(J)=W(J)+PS(J)*A(JA)                                       TPZ 770
            W(ML)=W(ML)+PS(K)*A(JA)                                     TPZ 780
  111    JA=JA+1                                                        TPZ 790
  112 CONTINUE                                                          TPZ 800
C                                                                       TPZ 810
C     IF N IS ODD, W(NMAX) HAS BEEN CALCULATED TWICE.                   TPZ 820
C                                                                       TPZ 830
      IF (N/2*2-N) 113,114,113                                          TPZ 840
  113 W(NMAX)=0.5*W(NMAX)                                               TPZ 850
  114 CONTINUE                                                          TPZ 860
C                                                                       TPZ 870
      DO 115 I=1,N                                                      TPZ 880
  115 W(I)=CINV*A(I)                                                    TPZ 890
      RETURN                                                            TPZ 900
C                                                                       TPZ 910
      END                                                               TPZ 920
```

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