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Covariance-Invariant Digital Filtering

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Abstract—When discretizing continuous-time filters, one is often interested in preserving a property termed covariance-invariance. Techniques are outlined for synthesizing discrete-time filters which are covariance-invariant with corresponding continuous-time filters. The synthesis techniques involve straightforward matrix decompositions or polynomial root-finding algorithms that can easily be programmed on a digital computer. Applications of the technique to digital filter synthesis are outlined, with example designs presented for covariance-invariant Butterworth and Chebyshev digital filters. Based on the frequency response of these designs it is argued that the method of covariance-invariance is superior to the methods of impulse-invariance and bilinear-z as a *response matching* design technique for the synthesis of digital filters. This superiority is especially apparent at sampling rates that are marginal with respect to filter critical frequencies. Moreover, the covariance-invariant designs are stably invertible solutions to a so-called spectral factorization problem. This property may be important in inverse filtering applications.

I. Introduction

RATHER AMAZINGLY, with all of the attention devoted to impulse-invariant, bilinear-z, and related discrete-time filter synthesis techniques, very little explicit attention has been devoted to the synthesis of discrete-time filters that are covariance-invariant with continuous-time filters. The property of covariance-invariance, to be carefully defined in Section II, ensures roughly that the covariance sequence associated with the output of a discrete-time filter excited by "white noise" equals the sampled covariance function associated with the sampled output of a continuous-time filter excited by "white noise." In this paper we examine covariance-invariance as a concept around which a response-fitting theory of digital filter synthesis may be constructed.

The concept of covariance-invariance is a statistical one, so it seems appropriate to classify covariance-invariant digital filters as statistically motivated or statistically designed digital filters. A similar classification applies to the classes of filters studied by Kellogg [1] and Farden and Scharf [2]. These investigators have achieved their design objectives indirectly by synthesizing filters to solve related minimum mean-squared error filtering problems. The covariance-invariance approach is rather more direct and classical: design objectives are achieved directly by equating a sampled covariance function with a covariance sequence. The result is a close match between analog and digital magnitude-squared frequency responses. The reasons for this are given in Section II of this paper. Other loosely related work has been reported by Greaves and Cadzow [3] and Chu *et al.* [4]. Greaves and Cadzow achieved a synthesis by minimizing the mean-squared error between two discrete-time sequences, one of which is the output of a digital filter and the other of which is the sampled output of an analog filter. The input to the analog filter is either a "rational" or "band-limited" covariance-stationary random process and the input to the digital filter is a sampled version of the same random process. Chu *et al.* have exploited results of Barlett [5] to establish the correspondence between the autoregressive (AR) and moving average (MA) parameters of an ARMA (2, 1) digital filter (2 poles, 1 zero) and the AR, or feedback, parameters of an all-pole second-order analog filter. The correspondence is achieved by matching the output covariance sequence of the ARMA (2, 1) filter to the sampled output covariance of the purely autoregressive analog filter. The ap-

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pearance of a zero in such covariance-invariant designs, even when the analog filter has no zero, was anticipated nearly thirty years ago by Bartlett [5] in a slightly different context.

Recently, a great deal of attention has been devoted to discrete-time spectral factorization for the synthesis of discrete-time systems with prescribed covariance sequences. See [6] for original contributions and a bibliography of other relevant work. When the prescribed covariance sequence corresponds to a sampled data sequence, then the problem is a covariance-invariant one. Mullis and Roberts [7] have discussed the use of first- *and* second-order information in the synthesis of discrete-time systems.

Frequency responses of Butterworth and Chebyshev digital filters designed by the methods of covariance-invariance, impulse-invariance, and bilinear-z are presented and compared in this paper. The results indicate a response-fitting superiority for the covariance-invariant designs, a superiority that may be important in applications that force the use of sampling rates that are only marginally higher than filter cutoff frequencies. We remark that the impulse-invariant design technique [8] is appropriate (within a scale constant) as a covariance-invariant design technique only for first-order filters, while the bilinear-z technique [9] is never appropriate because the required covariance-invariance is more demanding than the variance-preserving isomorphism established by Steiglitz for such designs [10]. There is a strong connection between covariance-invariant filter synthesis and spectral factorization, a connection discussed briefly in Section II and more fully in [11].

Finally, as discussed in [12], we emphasize that the issue of covariance-invariance also arises when simulating a discrete-time process that exhibits the covariance-invariant property with a related continuous-time process, and when designing a minimum mean-squared error (MMSE) estimator for a discrete-time process that has been obtained by sampling a continuous-time process. In the latter case, the covariance-invariant representation of the discrete-time data may be used to derive the appropriate sequential regression, or Kalman filter, equations provided the original continuous-time data are Markov.

The organization of this paper is as follows. In Secion II the frequency-domain synthesis of covariance-invariant digital filters is discussed. Section III contains example designs and comparisons with impulse-invariant and bilinear-z designs. In Section IV a time-domain technique for synthesizing multi-input covariance-invariant systems is discussed. Conclusions are advanced in Section V.

A preliminary version of this paper appeared in [12].

II. Covariance-Invariance

In our study of covariance-invariant digital filtering it is sufficient to consider stable single-input, single-output (SISO) continuous-time filters[1] characterized by rational transfer functions

$$H_c(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \quad m < n. \tag{1}$$

[1]More general multiinput, multioutput (MIMO) filters are considered in [11] and [12]. By stability we mean bounded-input, bounded-output (BIBO) stability.

Such a filter is termed an n-pole, m-zero analog filter. The spectrum of $H_c(s)$ is defined to be

$$S_c(s) = H_c(s)H_c(-s). \tag{2}$$

The covariance function $R_c(\tau)$ corresponding to $S_c(s)$ is obtained by inverse transforming $S_c(s)$:

$$R_c(\tau) = \frac{1}{2\pi j} \oint S_c(s)\, e^{s\tau}\, ds$$
$$S_c(s) = \int_{-\infty}^{\infty} R_c(\tau)\, e^{-s\tau}\, d\tau. \tag{3}$$

For $H_c(s)$ stable, $S_c(s)$ exists on the imaginary axis $s = j2\pi f$ and we may write

$$R_c(\tau) = \int_{-\infty}^{\infty} |H_c(s = j2\pi f)|^2\, e^{j2\pi f\tau}\, df. \tag{4}$$

The function $|H_c(s = j2\pi f)|^2$ is, of course, the magnitude-squared frequency response of $H_c(s)$. The covariance function $R_c(\tau)$ describes the covariance, $R_c(\tau) = Ey(t)y(t+\tau)$, of the steady-state filter output $y(t)$ when the system input $u(t)$ is a "white noise" process with covariance $Eu(t)u(t+\tau) = \delta(\tau)$. The notation E denotes expectation and $\delta(\tau)$ is the Dirac delta function.

Correspondingly, we are interested in stable SISO discrete-time filters characterized by rational transfer functions $H(z)$

$$H(z) = \frac{\beta_r z^{-r} + \beta_{r-1} z^{-r+1} + \cdots \beta_1 z^{-1} + \beta_0}{z^{-n} + \alpha_{n-1} z^{-n+1} + \cdots + \alpha_1 z^{-1} + \alpha_0}, \quad r < n. \tag{5}$$

This is termed an n-pole, r-zero digital filter. The spectrum of $H(z)$ is

$$S(z) = H(z)H(z^{-1}). \tag{6}$$

The covariance sequence is obtained by inverse transforming as follows:

$$R_k = \frac{1}{2\pi j} \oint S(z) z^{k-1}\, dz$$
$$S(z) = \sum_{k=-\infty}^{\infty} R_k z^{-k}. \tag{7}$$

For $H(z)$ stable, all poles of $H(z)$ lie within the unit circle $z = e^{j2\pi fT}$, $S(z)$ exists on the unit circle, and R_k may be written

$$R_k = T \int_{-1/2T}^{1/2T} |H(z = e^{j2\pi fT})|^2\, e^{j2\pi fkT}\, df. \tag{8}$$

The function $|H(z = e^{j2\pi fT})|^2$ is the periodic magnitude-squared frequency response of $H(z)$. The covariance sequence R_k describes the covariance, $R_k = Ey_l y_{l+k}$, of the steady-state filter output y_l when the system input u_l is a white-noise sequence with covariance $Eu_l u_{l+k} = \delta_k$. Here δ_k is the Kronecker delta.[2]

[2]We really should use the notation $\{R_k\}_{k=-\infty}^{\infty}$ and $\{u_k\}_{k=-\infty}^{\infty}$ when speaking of covariance sequences and input sequences; R_k is simply an entry in the sequence $\{R_k\}_{k=-\infty}^{\infty}$. However, the notation becomes a bit cumbersome so we largely avoid it and depend on the context to make the meaning clear.

Definition (Covariance-Invariance): Consider a SISO continuous-time filter with transfer function $H_c(s)$ and spectral representation $S_c(s)$. A discrete-time filter with transfer function $H(z)$ and spectral representation $S(z)$ is said to be *covariance-invariant* with the continuous-time filter if

$$R_k = R_c\,(\tau = kT)$$

for

$$k = 0, \pm 1, \pm 2 \cdots. \tag{9}$$

Comments: The property of covariance-invariance ensures that the covariance sequence characterizing the response of a stable, linear discrete-time system to "white noise" equals the sample covariance function that characterizes the sample output of a stable, linear continuous-time system excited by "white noise." Thus, a second-order statistical equivalence is established between samples of a continuous-time system output and a discrete-time system output. The need for such an equivalence arises naturally in MMSE filtering studies based on second-order statistics (of sampled data).

Using (7) and (9) the problem of synthesizing a covariance-invariant discrete-time system $H(z)$ can be phrased as one of finding the $H(z)$ such that

$$H(z)H(z^{-1}) = \sum_{k=-\infty}^{\infty} R_c(\tau = kT)z^{-k}. \tag{10}$$

In this form the covariance-invariant synthesis problem is one of spectrally factoring the right-hand side (RHS) of (10) and the relevant literature on minimal and partial realizations to achieve or approximate the solution of (10) may be brought to bear on the problem (for example [6] and [14]).

A. Why Covariance-Invariance for Frequency Response Matching?

The covariances R_k are Fourier series coefficients for the periodic function $|H(z = e^{j2\pi fT})|^2$. That is, R_k is the projection of $|H(z = e^{j2\pi fT})|^2$ onto $\{e^{j2\pi fkT}, -1/2T < f < 1/2T\}$. For $|H_c(s = j2\pi f)|^2$ essentially band-limited to $(-1/2T, 1/2T)$, indicating that $1/T$ has been chosen according to a Nyquist-like criterion, then $R_c(\tau = kT)$ is approximately equal to the projection of $|H_c(s = j2\pi f)|^2$ onto $\{e^{j2\pi fkT}, -1/2T < f < 1/2T\}$. By matching R_k to $R_c(\tau = kT)$ we are matching Fourier coefficients for $|H(z = e^{j2\pi fT})|^2$ with approximate Fourier coefficients for a periodic extension of $|H_c(s = j2\pi f)|^2$. By the unicity of Fourier series we are obtaining an approximate equality of $|H(z = e^{j2\pi fT})|^2$ and $|H_c(s = j2\pi f)|^2$ on the interval $(-1/2T, 1/2T)$.

This implicit matching of magnitude-squared responses, rather than the implicit matching of complex frequency response in the impulse-invariant and bilinear-z methods, accounts for the superior magnitude-squared response matching of covariance-invariant designs. A distinct advantage of the covariance-invariant designs is that they may always be chosen to be minimum-phase (or stably invertible) by simply choosing the zeros of $H(z)$ to be within the unit circle. This property is not enjoyed by impulse-invariant and bilinear-z designs, both of which are always nonminimum-phase. Of course, in an impulse-invariant design one may reflect zeros which lie outside the unit circle inside the unit circle without affecting the magnitude-squared frequency response of the filter; however, the impulse-invariant property will be destroyed. Experience with covariance-invariant designs has shown that nonminimum-phase covariance-invariant designs may provide better phase response matching than minimum-phase designs. Further, the phase response matching of the nonminimum-phase designs is approximately as accurate as it is for the impulse-invariant and bilinear-z designs. Thus when phase response matching is important, one may trade stable invertibility for accurate phase matching.

B. A Frequency-Domain Synthesis Procedure

The synthesis of a covariance-invariant digital filter proceeds from the expression for $R_c(\tau)$ given in (3) or (4) and the spectral factorization formula of (10). That is, given the continuous-time transfer function $H_c(s)$, one obtains $R_c(\tau)$ from (3) and samples it to obtain $R_l = R_c(\tau = lT)$. The sequence $\{R_l, l = 0, \pm 1, \cdots\}$ is z-transformed to obtain the RHS of (10), which is then factored into $H(z)$ and $H(z^{-1})$. To illustrate, consider the SISO continuous-time filter with transfer function

$$H_c(s) = \sum_{i=1}^{n} \frac{A_i}{s + s_i} \tag{11}$$

where A_i and s_i, $i = 1, 2, \cdots, n$, may be complex and $\mathrm{Re}\,\{s_i\} > 0$ for $i = 1, 2, \cdots, n$. When this continuous-time filter is excited by a white process of covariance $\delta(\tau)$, then the steady-state covariance of the filter output is obtained from the inversion of (4):

$$R_c(\tau) = \sum_{i=1}^{n} A_i\, e^{-s_i|\tau|} H_c(s_i). \tag{12}$$

It follows that (10) may be written

$$H(z)H(z^{-1}) = \sum_{k=-\infty}^{\infty} R_c(\tau = kT)z^{-k}$$

$$= z \sum_{i=1}^{n} A_i\, H_c(s_i)\, \frac{e^{-s_iT} - e^{s_iT}}{(z - e^{-s_iT})(z - e^{s_iT})}. \tag{13}$$

The RHS of (13) exists for $|z_1^{-1}| < |z| < |z_1|$ with $z_1 \triangleq \min_i |e^{s_iT}| > 1$. Equation (13) can be written as the following ratio of polynomials in z:

$$H(z)H(z^{-1}) = \frac{N(z)}{\prod_{i=1}^{n} (z - e^{-s_iT})(z^{-1} - e^{-s_iT})} \tag{14}$$

where the numerator polynomial $N(z)$ is of order $2(n - 1)$ in z:

$$N(z) = \sum_{i=1}^{n} A_i\, H_c(s_i)(1 - e^{-2s_iT}) \prod_{\substack{k=1 \\ k \neq i}}^{n} (z - e^{-s_kT})(z^{-1} - e^{-s_kT}). \tag{15}$$

The factorization to obtain $H(z)$ proceeds by choosing distinct poles $z = e^{-s_iT}$ (these are simply the impulse-invariant poles) and then solving for the $2(n - 1)$ roots of $N(z)$. One is then free to associate the $n - 1$ roots that lie inside the unit circle with $H(z)$, and the $(n - 1)$ roots that lie outside with $H(z^{-1})$, thereby obtaining a minimum phase design.

We have essentially outlined the proof of the following theorem.

Theorem: Given an nth-order stable SISO continuous-time filter $H_c(s)$ with distinct poles s_i, $i = 1, 2, \cdots, n$ and $m < n$ zeros. The corresponding stable SISO covariance-invariant digital filter has n poles (the impulse-invariant poles) and $(n - 1)$ zeros.

Comment: This theorem is proved and extended to include continuous-time filters $H_c(s)$ with repeated poles in [11]. A similar idea is stated by Jenkins and Watts [15]: "Whereas the original continuous AR process has an input which is white noise, the discrete sampled AR process has an input which is an MA process of order one less than the order of the differential equation describing the system." This statement is not proved, but a reference is given to Bartlett [5] who proved it constructively for a second-order case. A general proof is contained in [11].

C. Synthesis Procedure: A Summary

The design steps in the synthesis are the following.

1) Evaluate $R_c(\tau)$ from (3) (or by correlation if the impulse response $h(t)$ is known).

2) Evaluate the RHS of (10).

3) Factor the RHS of (10) by choosing impulse-invariant poles and solving for the $2(n - 1)$ zeros of $N(z)$, to obtain the transfer function $H(z)$.

Comments: Whenever $R_c(\tau)$ is known, as in process simulation applications, then only the last two design steps are required. In step 3), which is a spectral factorization problem, relevant results from a vast literature may be used in the factorization. When only a finite record of $R_c(\tau)$ is given, the so-called partial realization problem is relevant and numerous approximation algorithms are to to found in the literature to solve the factorization [16].

An important observation in the design of covariance-invariant discrete-time filters is that the gain due to sampling is $1/\sqrt{T}$ instead of $1/T$ as usual.

III. Example Designs

In order to demonstrate the excellent frequency response fitting properties of covariance-invariant digital filters, we present several example designs for a variety of Chebyshev and Butterworth covariance-invariant filters.

Consider the continuous-time fourth-order Butterworth low-pass filter with $\omega_c = 1$ rad/s, whose transfer function has the following poles:

$$s_{1;2} = 0.3827 \pm j\,0.9239$$
$$s_{3;4} = 0.9239 \pm j\,0.3827. \tag{16}$$

For $T = 0.1$ s, the synthesis procedure of Section II results in a discrete-time covariance-invariant filter with the following poles:

$$p_{1;2} = 0.9583 \pm j\,0.0888$$
$$p_{3;4} = 0.9111 \pm j\,0.0349. \tag{17}$$

The numerator polynomial of (15) is

$$N(z) = [3.6913 + 1.8197(z + z^{-1}) + 1.8334 \cdot 10^{-1}(z^2 + z^{-2}) + 1.5274 \cdot 10^{-3}(z^3 + z^{-3})] \cdot 10^{-8} \tag{18}$$

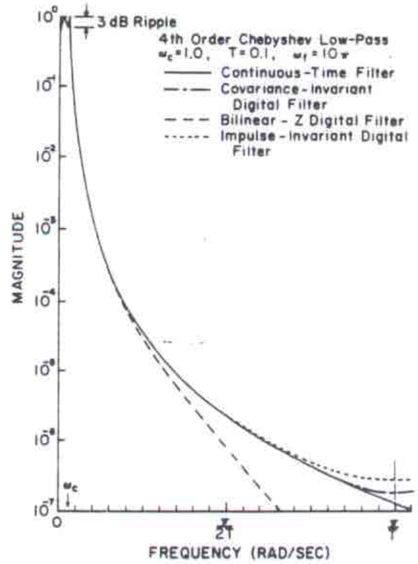

Fig. 1. Magnitude responses for Chebyshev designs; $T = 0.1$.

with roots

$$z_1 = -0.1226 \qquad z_4 = -8.1591$$
$$z_2 = -0.5353 \qquad z_5 = -1.8682 \tag{19}$$
$$z_3 = -0.0091 \qquad z_6 = -109.3434.$$

As expected from a symmetric polynomial, $z_1 = z_4^{-1}$, $z_2 = z_5^{-1}$, and $z_3 = z_6^{-1}$. Thus choosing the zeros which are within the unit circle, the minimum-phase, fourth-order low-pass covariance-invariant discrete-time Butterworth filter has the following transfer function:

$$H(z) = \frac{1.5955 \cdot 10^{-4}(z + 0.1226)(z + 0.5353)(z + 0.0091)}{[(z - 0.9583)^2 + 0.0888^2]\,[(z - 0.9111)^2 + 0.0349^2]}. \tag{20}$$

The frequency response of (20) is presented in Figs. 6 and 7 together with the frequency response of $H_c(s)$ and the frequency responses of two other digital filters: one designed by the impulse-invariant procedure and the second by the bilinear-z procedure. It is evident from Fig. 6 that the covariance-invariant digital filter has better magnitude response fitting properties than the other two. This superiority is more remarkable for sampling rates T^{-1} that are close to the critical frequencies of the filter (e.g., Figs. 11, 13). In Fig. 8 the phase response of the minimum-phase covariance-invariant design is compared to the phase responses of two (out of seven possible) nonminimum-phase covariance-invariant designs. From Figs. 7 and 8 and Figs. 3 and 4, the phase response matching of nonminimum-phase covariance-invariant designs is approximately as accurate as it is for impulse-invariant and bilinear-z designs.

A variety of similar covariance-invariant designs for Butterworth and Chebyshev filters are summarized in Tables I and II and in Figs. 1-13. In all figures results are given for impulse-invariant designs, prewarped bilinear-z designs, and covariance-invariant designs. In Fig. 2 an extra magnitude response is plotted to illustrate the effect of simply cascading two second-order covariance-invariant digital filters rather than solving for the fourth-order covariance-invariant digital filter.

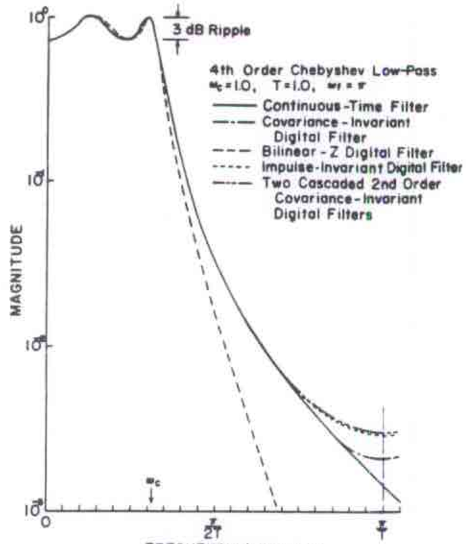

Fig. 2. Magnitude responses for Chebyshev designs; $T = 1.0$.

In all examples considered to date the frequency response matching of the covariance-invariant design is superior to the corresponding magnitude response matching for the impulse-invariant and bilinear-z designs. For sampling rates T^{-1} that are marginal with respect to filter critical frequencies, as in Figs. 11 and 13, the magnitude response matching properties of the covariance-invariant designs are far superior to the impulse-invariant and bilinear-z designs. The pole-zero diagrams of Figs. 5 and 9 illustrate the differences in pole and zero locations for impulse-invariant, bilinear-z, and covariance-invariant designs. The solid dots represent fourth-order zeros for the bilinear-z design. As indicated in the diagrams the only distinction between the covariance-invariant designs and the impulse-invariant designs is in the location of the zeros. This systematic relocation of zeros yields improved magnitude response matching and stable-invertibility.

IV. MISO Synthesis Procedure

The synthesis procedure of Section II is a straightforward frequency-domain procedure. In this section we discuss a time-domain approach to covariance-invariant synthesis that is applicable to multiinput, single-output (MISO) discrete-time systems. Such systems are not applicable to SISO digital filtering, but they are useful for the synthesis of random sequences that are covariance-invariant with randomly sampled data sequences.

Our convention in Section IV is that a state model for a random sequence is always excited by a white sequence. Most of the material on state equations in this section can be found in [13, ch. 6].

A. Continuous-Time State Equations

There are a variety of state-space representations of (1). The representation which leads to the most direct calculation of covariance-invariant equations is the so-called Luenberger canonical form (F, G, C) with

$$\dot{X}(t) = FX(t) + Gu(t)$$
$$y(t) = CX(t) \tag{21}$$

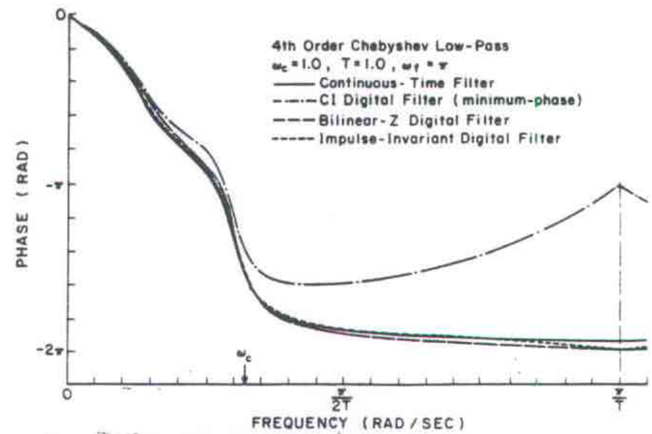

Fig. 3. Phase responses for Chebyshev low-pass designs; $T = 1.0$.

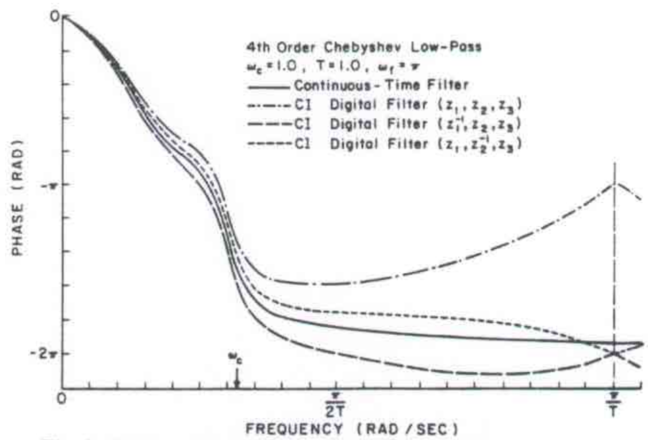

Fig. 4. Phase responses for Chebyshev low-pass designs; $T = 1.0$.

and

$$\underset{(n\times n)}{F} = \left[\begin{array}{c|c} 0 & I \\ \hline -a_0 & -a_1 \cdots -a_{n-1} \end{array}\right]; \quad \underset{(1\times n)}{C} = [1\;0\cdots 0]$$

$$\underset{(n\times 1)}{G} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \qquad \begin{aligned} g_1 &= b_{n-1} - b_n a_{n-1} \\ g_2 &= b_{n-2} - b_n a_{n-2} - g_1 a_{n-1} \\ &\;\vdots \\ g_{n-k} &= b_k - \sum_{i=0}^{n-k-1} g_i a_{i+k} \\ g_n &= b_n - \sum_{i=0}^{n-1} g_i a_i \end{aligned} \tag{22}$$

where we define $b_i = 0$ for $i = m+1, m+2, \cdots, n-1$. In (21), $y(t)$ is the system output and $u(t)$ is the system input. The transfer function $H_c(s)$ is given by

$$H_c(s) = C(sI - F)^{-1}G. \tag{23}$$

This transfer function exists for $\text{Re}\{s\} > \max_i \text{Re}\{\lambda_i(F)\}$, where $\lambda_i(F)$ is the ith eigenvalue of F and $\text{Re}\{\lambda_i(F)\} < 0$ $(1 \leq i \leq n)$.

When $u(t)$ in (21) is a "white" process, then the steady-state covariance of $y(t)$ can be written

$$R_c(\tau) = \begin{cases} Ce^{-F\tau}V_cC', & \tau < 0 \\ CV_c e^{F'\tau}C', & \tau \geq 0. \end{cases} \tag{24}$$

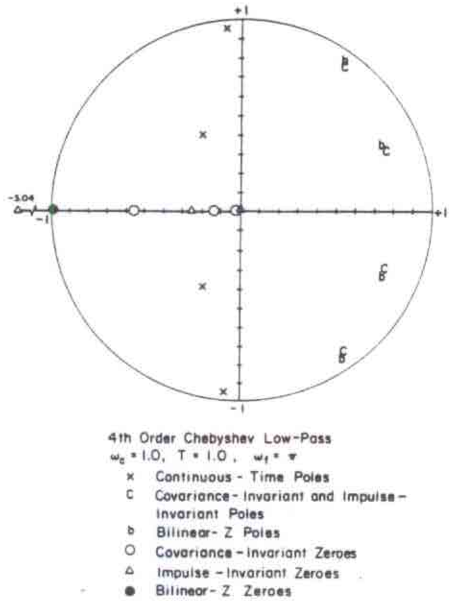

Fig. 5. Pole-zero locations for Chebyshev designs; $T = 1.0$.

Here V_c is the steady-state zero-lag covariance matrix for the state $X(t)$. This covariance matrix satisfies the Lyapunov equation

$$FV_c + V_cF' + GG' = 0 \tag{25}$$

or the integral equation

$$V_c = \int_0^{\infty} e^{Ft}GG'\, e^{F't}\, dt \tag{26}$$

where superscript prime denotes matrix transpose.

B. Discrete-Time State Equations

A state-space model for (5) that is useful in the synthesis of multiinput covariance-invariant systems is

$$\begin{aligned} X_{k+1} &= \Phi X_k + \Gamma U_k \\ y_k &= \Psi X_k. \end{aligned} \tag{27}$$

Here Φ is an $(n \times n)$ state-transition matrix, Γ is an $(n \times n)$ input matrix, Ψ is the $(1 \times n)$ output vector $[1\ 0 \cdots 0]$, and U_k is an $(n \times 1)$ white input *vector*. This is a state-model for a MISO discrete-time system.

For U_k a white *vector*, i.e., $EU_kU_{k+l} = I\delta_l$, the covariance of y_k is given by

$$R_l = \begin{cases} \Psi\Phi^{-l}V_d\Psi', & l < 0 \\ \Psi V_d(\Phi')^l\Psi', & l \geq 0 \end{cases} \tag{28}$$

with V_d satisfying the equation

$$V_d = \Phi V_d \Phi' + \Gamma\Gamma'. \tag{29}$$

Of course R_l will not be the correct model for the system of (5) when U_k is chosen to be $U_k = (u_k, u_{k-1}, \cdots, u_{k-n})$. The reason is that white u_k results in a nonwhite covariance for U_k; that is $EU_kU'_{k+l} = \Omega_l$ where Ω_l is a matrix with ones on the lth diagonal. The main diagonal is the zeroth diagonal.

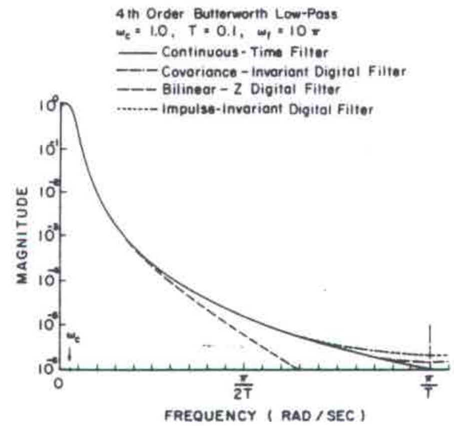

Fig. 6. Magnitude responses for Butterworth designs; $T = 0.1$.

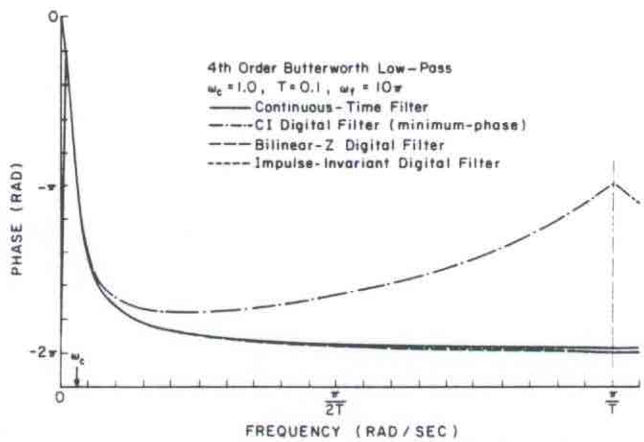

Fig. 7. Phase responses for Butterworth low-pass designs; $T = 0.1$.

C. MISO Synthesis

The appropriate MISO synthesis equations are (24), (25), (28), and (29). From (24) the covariance $R_c(\tau)$ is computed. The sampled version is simply

$$R_c(\tau = kT) = \begin{cases} C\, e^{-FkT}V_cC', & k < 0 \\ CV_c\, e^{F'kT}C', & k \geq 0 \end{cases} \tag{30}$$

with V_c given by (25). For MISO synthesis, in which case a white vector U_k may be employed, one may consider the covariance model of (28). Then a covariance-invariant design is achieved by equating $R_c(\tau = kT)$ with the expression given for R_k in (28). The result is

$$\begin{aligned} &1)\ \Psi = C \\ &2)\ \Phi = e^{FT} \\ &3)\ V_c = V_d. \end{aligned} \tag{31}$$

Condition 3) imposes a solution constraint on the *matrix* Γ via (29):

$$V_c = \Phi V_c \Phi' + \Gamma\Gamma. \tag{32}$$

The following theorem ensures that a solution for Γ exists.

Theorem: The $(n \times n)$ matrix $Q(T) \triangleq V_c - e^{FT}V_c\, e^{F'T}$ is symmetric and nonnegative definite for all $T \in [0, \infty)$.

TABLE I
CHEBYSHEV FILTER DESIGNS

Synthesis Method / Filter Type	H(z), covariance-invariant design	H(z), impulse-invariant design	H(z), bilinear z-transform design
4th order Chebyshev L.P. filter $\omega_c=1$; T=0.1 Cf. Fig. 1	$\frac{2.2119\cdot10^{-5}(z+0.1231)(z+0.5345)(z+0.0090)}{[(z-0.9871)^2+0.0937^2][(z-0.9789)^2+0.0384^2]}$	$\frac{2.0575\cdot10^{-5}\cdot z(z+3.6757)(z+0.2643)}{[(z-0.9871)^2+0.0937^2][(z-0.9789)^2+0.0384^2]}$	$\frac{7.6145\cdot10^{-7}\ (z+1)^4}{[(z-0.9871)^2+0.0937^2][(z-0.9789)^2+0.0384^2]}$
4th order Chebyshev L.P. filter $\omega_c=1$; T=1.0 Cf. Figs. 2-5	$\frac{4.8552\cdot10^{-2}(z+0.1321)(z+0.5523)(z+0.0098)}{[(z-0.5368)^2+0.7451^2][(z-0.7524)^2+0.3111^2]}$	$\frac{1.7126\cdot10^{-2}\cdot z(z+3.0368)(z+0.2466)}{[(z-0.5368)^2+0.7451^2][(z-0.7524)^2+0.3111^2]}$	$\frac{6.3843\cdot10^{-3}(z+1)^4}{[(z-0.5361)^2+0.7589^2][(z-0.7337)^2+0.3339^2]}$
4th order Chebyshev B.P. filter $\omega_o=\sqrt{40}$;BW=3;T=0.1 Cf. Fig. 10	$\frac{9.8439\cdot10^{-2}(z+0.2950)[(z-0.9457)^2+0.0530^2]}{[(z-0.8319)^2+0.4813^2][(z-0.6857)^2+0.6494^2]}$	$\frac{3.5182\cdot10^{-1}\cdot z(z-1.1325)(z-0.8893)}{[(z-0.8319)^2+0.4813^2][(z-0.6857)^2+0.6494^2]}$	$\frac{1.0281\cdot10^{-2}(z+1)^2(z-1)^2}{[(z-0.8307)^2+0.4816^2][(z-0.6857)^2+0.6515^2]}$
4th order Chebyshev B.P. filter $\omega_o=\sqrt{40}$;BW=3;T=0.39 Cf. Fig. 11	$\frac{3.0449\cdot10^{-1}(z+0.7869)[(z-0.1571)^2+0.3834^2]}{[(z+0.3918)^2+0.7622^2][(z+0.7865)^2+0.1466^2]}$	$\frac{-4.6110\cdot10^{-1}\cdot z(z+1.3739)(z+0.3766)}{[(z+0.3918)^2+0.7622^2][(z+0.7865)^2+0.1466^2]}$	$\frac{1.2656\cdot10^{-1}(z+1)^2(z-1)^2}{[(z+0.3859)^2+0.6074^2][(z+0.9903)^2+0.0236^2]}$

TABLE II
BUTTERWORTH FILTER DESIGNS

Synthesis Method / Filter Type	H(z), covariance-invariant design	H(z), impulse-invariant design	H(z), bilinear z-transform design
4th order Butterworth L.P. filter $\omega_c=1$; T=0.1 Cf. Figs. 6-8	$\frac{1.5955\cdot10^{-4}(z+0.1226)(z+0.5353)(z+0.0091)}{[(z-0.9583)^2+0.0888^2][(z-0.9111)^2+0.0349^2]}$	$\frac{1.5608\cdot10^{-4}\cdot z(z+3.4937)(z+0.2512)}{[(z-0.9583)^2+0.0888^2][(z-0.9111)^2+0.0349^2]}$	$\frac{5.4847\cdot10^{-6}(z+1)^4}{[(z-0.9584)^2+0.0888^2][(z-0.9111)^2+0.0350^2]}$
4th order Butterworth L.P. filter $\omega_c=1$; T=0.5 Cf. Fig. 9	$\frac{2.6446\cdot10^{-2}(z+0.1226)(z+0.5353)(z+0.0091)}{[(z-0.7393)^2+0.3681^2][(z-0.6186)^2+0.1198^2]}$	$\frac{1.4862\cdot10^{-2}\cdot z(z+2.6601)(z+0.1958)}{[(z-0.7393)^2+0.3681^2][(z-0.6186)^2+0.1198^2]}$	$\frac{2.0436\cdot10^{-3}(z+1)^4}{[(z-0.7477)^2+0.3684^2][(z-0.6150)^2+0.1255^2]}$
4th order Butterworth B.P. filter $\omega_o=3\sqrt{10}$; BW=1; T=0.03 Cf. Fig. 12	$\frac{4.0065\cdot10^{-3}(z+0.2729)[(z-0.9839)^2+0.0159^2]}{[(z-0.9529)^2+0.2678^2][(z-0.9463)^2+0.2878^2]}$	$\frac{2.8576\cdot10^{-2}\cdot z(z-1)^2}{[(z-0.9529)^2+0.2678^2][(z-0.9463)^2+0.2878^2]}$	$\frac{2.2031\cdot10^{-4}(z+1)^2(z-1)^2}{[(z-0.9529)^2+0.2678^2][(z-0.9463)^2+0.2878^2]}$
4th order Butterworth B.P. filter $\omega_o=3\sqrt{10}$;BW=1;T=0.3 Cf. Fig. 13	$\frac{4.9371\cdot10^{-2}(z+0.2249)(z+0.8742)(z+0.0381)}{[(z+0.8311)^2+0.3529^2][(z+0.8798)^2+0.1687^2]}$	$\frac{-1.1779\cdot10^{-1}\cdot z(z+1.1849)(z+0.6490)}{[(z+0.8311)^2+0.3529^2][(z+0.8798)^2+0.1687^2]}$	$\frac{1.8472\cdot10^{-2}(z+1)^2(z-1)^2}{[(z+0.8047)^2+0.3058^2][(z+0.9294)^2+0.1384^2]}$

Proof: The symmetry of $Q(T)$ follows from the symmetry of V_c. Define

$$Q(t) = V_c - e^{Ft}V_c\, e^{F't}, \qquad Q(0) = 0$$
$$\dot{Q}(t) = e^{Ft}GG'\, e^{F't}, \qquad \dot{Q}(0) = GG' \geqslant 0. \tag{33}$$

Thus $\dot{Q}(t)$ is nonnegative definite and $Q(T) = \int_0^T \dot{Q}(t)$ is nonnegative definite for $T \in [0, \infty)$.

Comment: The nonnegative definiteness of $Q(T)$ ensures that the "variance equation" (32) can be solved by a Cholesky decomposition algorithm. We emphasize that the resulting discrete-time system (Φ, Γ, Ψ) is a multiinput system excited by a vector of uncorrelated components with the property $EU_kU'_{k+l} = I\delta_l$.

D. Numerical Example

Consider a second-order SISO continuous-time system with input dynamics and transfer function

$$H_c(s) = \frac{s + 7.5}{(s + 0.5)(s + 4)}. \tag{34}$$

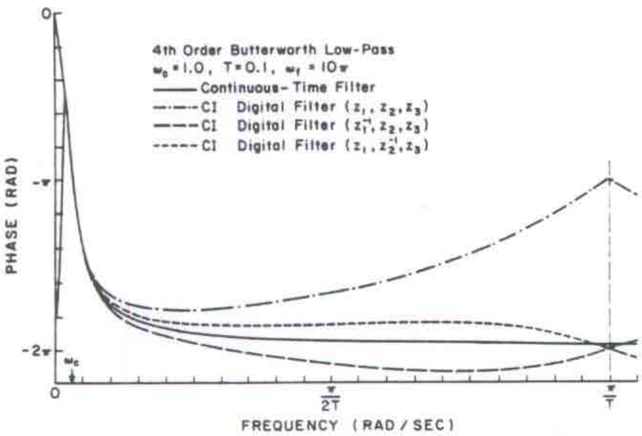

Fig. 8. Phase responses for Butterworth low-pass designs; $T = 0.1$.

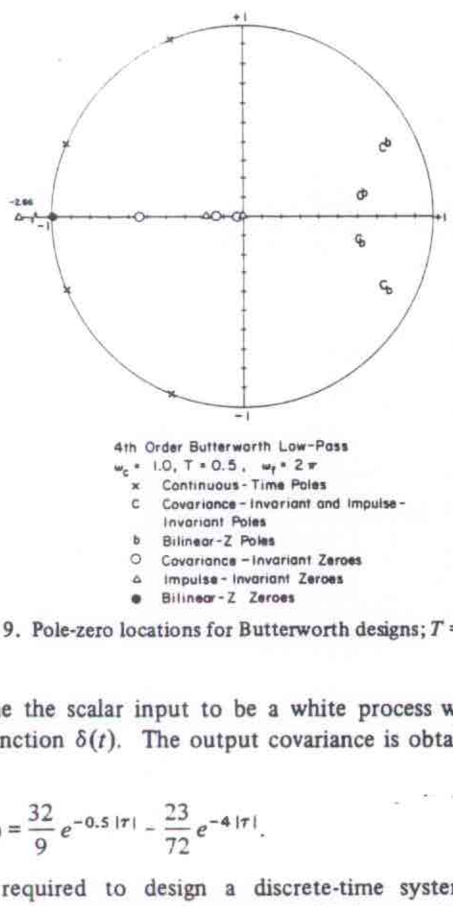

Fig. 9. Pole-zero locations for Butterworth designs; $T = 0.5$.

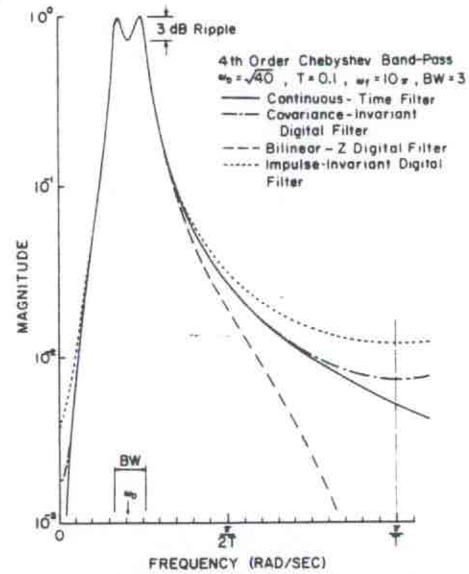

Fig. 10. Magnitude responses for Chebyshev designs; $T = 0.1$.

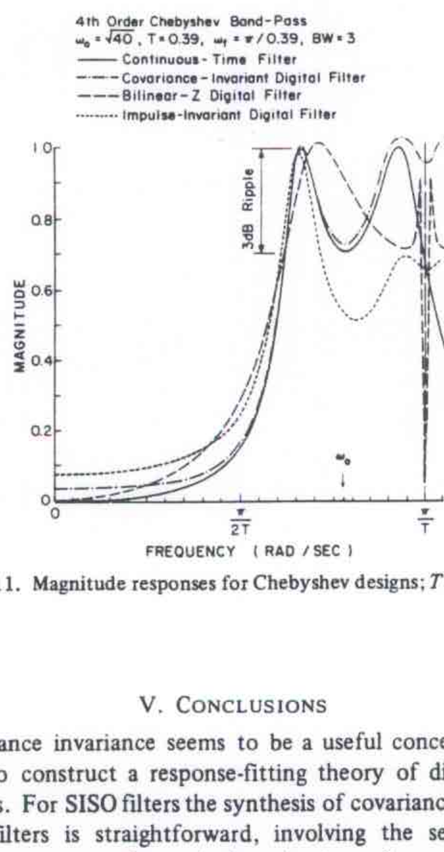

Fig. 11. Magnitude responses for Chebyshev designs; $T = 0.39$.

Assume the scalar input to be a white process with covariance function $\delta(t)$. The output covariance is obtained from (12):

$$R_c(\tau) = \frac{32}{9} e^{-0.5|\tau|} - \frac{23}{72} e^{-4|\tau|}. \tag{35}$$

It is required to design a discrete-time system that is covariance-invariant with the given $H_c(s)$, for sampling interval $T = 0.1$ s. The frequency-domain synthesis procedure results in the following covariance-invariant discrete-time SISO system

$$H(z) = \frac{0.3572(z - 0.4662)}{(z - 0.9512)(z - 0.6703)}. \tag{36}$$

The time-domain synthesis procedure results in the following MISO covariance-invariant discrete-time system (Φ, Γ, Ψ):

$$\Phi = \begin{bmatrix} 0.9914 & 0.0803 \\ -0.1605 & 0.6302 \end{bmatrix}; \quad \Gamma = \begin{bmatrix} 0.3569 & 0 \\ 0.7259 & 0.1590 \end{bmatrix}; \quad \Psi = [1 \quad 0]. \tag{37}$$

V. Conclusions

Covariance invariance seems to be a useful concept around which to construct a response-fitting theory of digital filter synthesis. For SISO filters the synthesis of covariance-invariant digital filters is straightforward, involving the selection of impulse-invariant poles and the selection of zeros that are roots of a $2(n - 1)$-order polynomial in z. For high-order filters one can simply use standard polynomial root finding algorithms available in scientific subroutine packages. The designs presented in this paper suggest that covariance-invariant digital filters are only slightly more difficult to design than their impulse-invariant and bilinear-z transform counterparts, and that they outperform them as magnitude-squared frequency response matching filters.

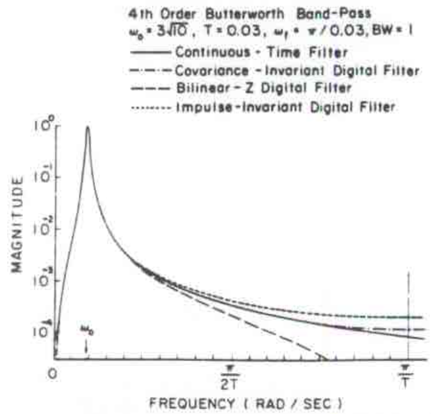

Fig. 12. Magnitude responses for Butterworth designs; $T = 0.03$.

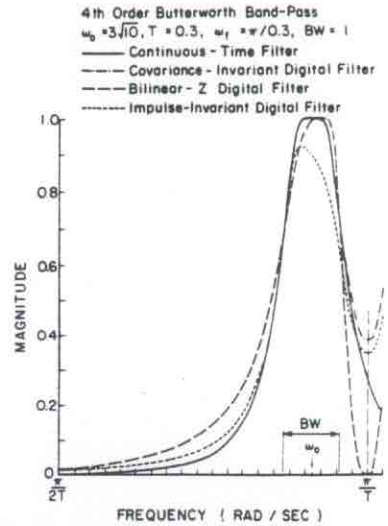

Fig. 13. Magnitude responses for Butterworth designs; $T = 0.3$.

The results on MISO synthesis in Section IV are useful for the synthesis, using independent random numbers, of random sequences having a prescribed covariance function. When the independent random numbers are Gaussian, then so is the synthesized sequence. When using the MISO time-domain synthesis method, one simply employs a Cholesky decomposition algorithm to solve for the input matrix Γ. The theorem of Section IV ensures a solution exists.

Finally, we remark that the parallel or serial cascades of low-order discrete-time sections that are covariance-invariant with corresponding low-order continuous-time sections are not covariance-invariant. Thus pole-grouping, which is useful in the design of high-order digital filters, cannot be directly applied in covariance-invariant digital filter design. The same difficulty arises in impulse-invariant digital filtering. Only in the bilinear-z transform design method is it possible to cascade and to parallel digital filters, and obtain the correct overall digital filter. If, however, in the covariance-invariant design of cascaded filters it is taken into consideration that the input spectrum of each filter is the output spectrum of the filter preceding it, then covariance-invariant digital filters can be designed which will result in the desired overall covariance-invariant filter.

Acknowledgment

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