

Colorado State University, Ft. Collins

ECE 421: Telecommunications I  
Fall Semester, 2008

Midterm Exam  
Oct. 14, 2008  
2:00pm-3:15pm, Engr B103

Name: Problem Solutions

Problem 1 (28 points) \_\_\_\_\_

Problem 2 (42 points) \_\_\_\_\_

Problem 3 (30 points) \_\_\_\_\_

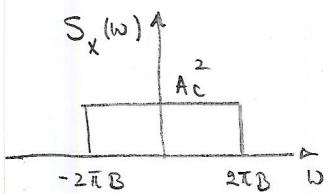
Total: (100) \_\_\_\_\_

### Problem 1 (28 Points)

**Part A (12 points).** Let  $z(t) = x(t) \cos^2(w_c t)$ , where  $x(t)$  is a power signal with power spectral density  $S_x(\omega) = A_c^2$  for  $|\omega| \leq 2\pi B$  and  $S_x(\omega) = 0$  for  $|\omega| > 2\pi B$ . Assume  $2\pi B \ll w_c$ . Sketch the power spectral density  $S_z(\omega)$  of  $z(t)$  and find the power of  $z(t)$ .

**Part B (16 points).** For the  $z(t)$  defined in Part A, let  $u(t) = z(t) \text{sinc}(Bt)$ . Sketch the power spectral density  $S_u(\omega)$  and find the power of  $u(t)$  and the smallest delay  $\tau$  where the autocorrelation of  $u(t)$ , i.e.  $R_u(\tau)$ , is zero.

Part A :



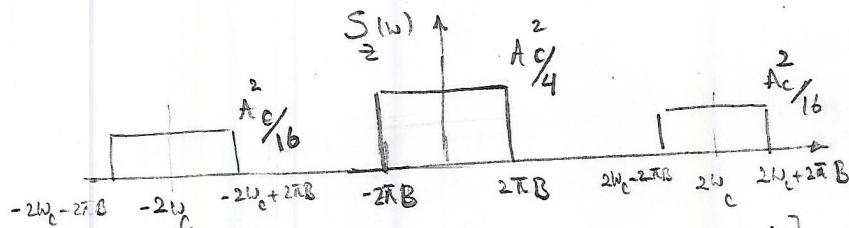
$$S_x(\omega) = |X(\omega)|^2$$

$$z(t) = x(t) \cos^2(w_c t) = \frac{1}{2} x(t) (1 + \cos 2w_c t)$$

$$Z(\omega) = \frac{1}{2} X(\omega) + \frac{1}{4} X(\omega - w_c) + \frac{1}{4} X(\omega + w_c)$$

$$S_z(\omega) = |Z(\omega)|^2 \quad \xrightarrow{w_c \gg 2\pi B}$$

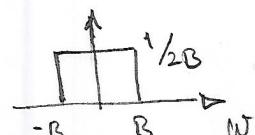
$$S_z(\omega) = \frac{1}{2} S_x(\omega) + \frac{1}{16} [S_x(\omega - 2w_c) + S_x(\omega + 2w_c)]$$



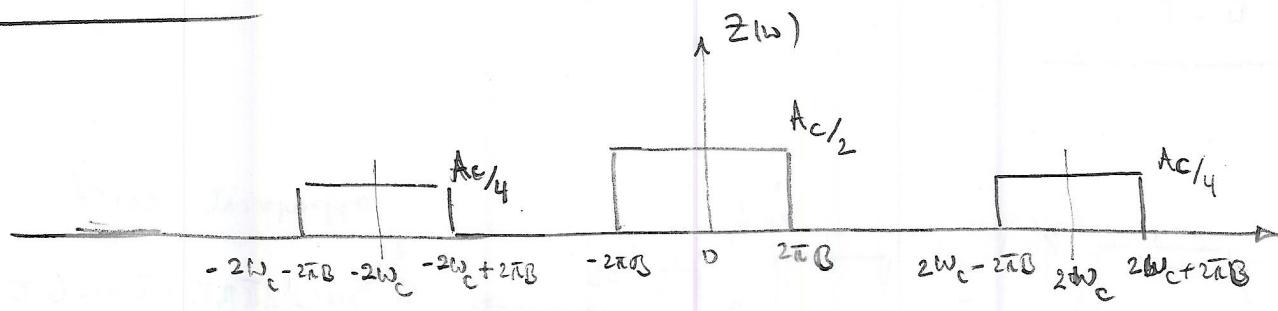
$$\begin{aligned} P_z &= \frac{1}{2\pi} \int S_z(\omega) d\omega = \frac{1}{2\pi} \left[ 2 \times 4\pi B \times \frac{A_c^2}{16} + 4\pi B \times \frac{A_c^2}{4} \right] \\ &= \underline{\underline{\frac{3}{4} B A_c^2}} \end{aligned}$$

Part B :

$$\begin{aligned} u(t) &= z(t) \text{sinc}(Bt) \leftrightarrow U(\omega) = \frac{1}{2\pi} Z(\omega) * \frac{2\pi}{2B} \text{rect}\left(\frac{\omega}{2B}\right) \\ &= Z(\omega) * \frac{1}{2B} \text{rect}\left(\frac{\omega}{2B}\right) \end{aligned}$$

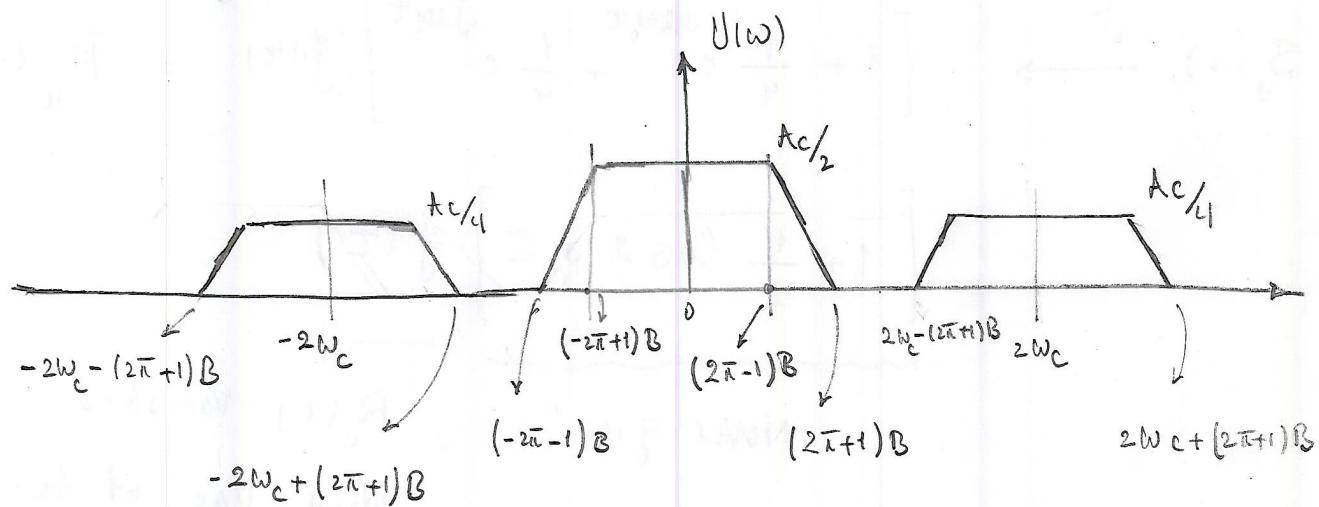


Part B - Cont.



$\frac{1}{2}B$  → Slide and calculate  
area under intersection

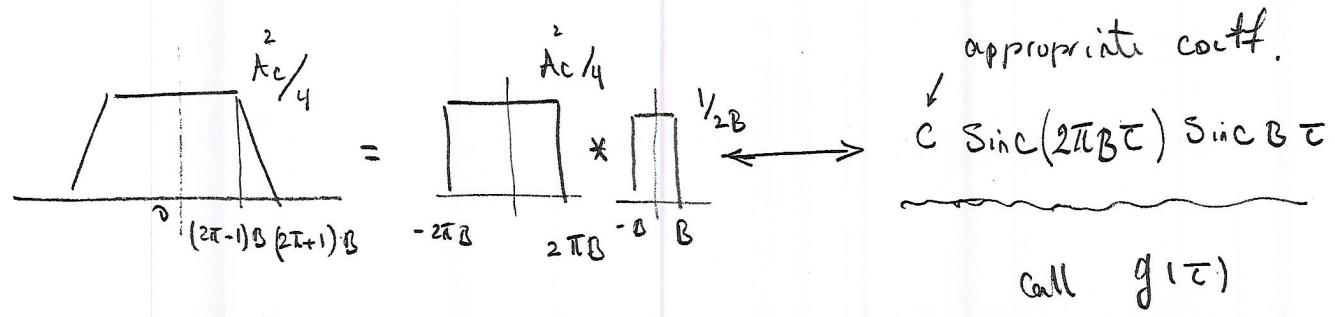
$w-B \quad w \quad w+B$



$$S_u(\omega) = |U(\omega)|^2 \xrightarrow{w_c > 2\pi B} \text{Same shape with squared amplitudes.}$$

$$\begin{aligned} P_u &= \frac{1}{2\pi} \int S_u(\omega) d\omega \\ &= \frac{1}{2\pi} \left[ \underbrace{\left( \frac{A_c^2}{4} \right) \times \left[ (4\pi-2)B + \frac{2}{2} \times 2B \right]}_{\text{area under middle trapezoid}} \left( 1 + \frac{1}{4} + \frac{1}{4} \right) \right] \\ &\quad \text{Centered at } 0 \quad \text{Centered at } 2\omega_c \quad \text{centered at } -2\omega_c \\ &= \underbrace{\frac{3A_c^2}{16\pi} 4\pi B}_{4} = \frac{3}{4} A_c^2 B \end{aligned}$$

Part B - Cont.



$$S_U(\omega) \xrightarrow{\mathcal{F}^{-1}} \left[ 1 + \frac{1}{4} e^{j2\omega_c \tau} + \frac{1}{4} e^{-j2\omega_c \tau} \right] g(\tau) = R_u(\tau)$$

$$\left[ 1 + \frac{1}{2} \cos 2\omega_c \tau \right] g(\tau)$$

Never zero

$R_u(\tau)$  vanishes  
when one of the  
sincs is zero.

$\text{sinc}(2\pi B \tau)$  varies faster than  $\text{sinc} B \tau$   
so it goes to zero first.

$$\text{sinc}(2\pi B \tau) = 0 \rightarrow \tau = 1/2B$$

## Problem 2 (42 Points)

Consider the frequency modulation (FM) of the message signal

$$m(t) = -\cos(200\pi t) + \sin(50\pi t)$$

Suppose the base carrier frequency is  $f_c = 1000$  Hz, the FM parameter is  $k_f$ , and the carrier amplitude is  $A$ .

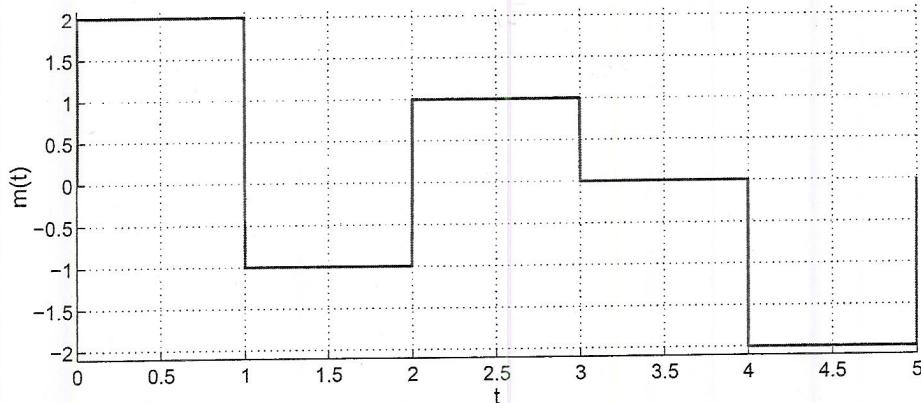
**Part A (8 points).** Give the FM waveform.

**Part B (8 points).** What is the frequency deviation of the FM signal?

**Part C (8 points).** Select a value for  $k_f$  that will achieve a Carlson's rule bandwidth of 300 Hz for FM.

**Part D (8 points).** What value for the carrier amplitude  $A$  will produce a 50 Watt output power in a  $1\Omega$  system?

Now Consider the phase modulation (PM) of the following message signal:



**Part E (10 points).** Give a simplified expression for the corresponding PM signal when the PM parameter is  $k_p = \pi$ , the carrier amplitude is  $A = 1$ , and the base carrier frequency is  $f_c = 4$  Hz. Will there be any ambiguity in decoding the signal? If yes, what are the ambiguities?

Part A:  $a_{ft} = \int [-\cos(200\pi t) + \sin(50\pi t)] dt = -\frac{\sin 200\pi t}{200\pi} - \frac{\cos 50\pi t}{50\pi}$

$$\phi_{PM}(t) = A \cos(200\pi t + K_p a_{ft})$$

Part B:  $\Delta f = \frac{K_f M_p}{2\pi}, \quad M_p = \max |m(t)|$

$$\max |\cos 200\pi t| = 1 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Here they can be achieved at the same time at } \\ \max |\sin 50\pi t| = 1 \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad t = \frac{3}{100}, \rightarrow m_p = 2$$

$$\Delta f = \frac{K_f m_p}{2\pi} = \underline{\underline{\frac{K_f}{\pi}}}$$

Part C :

$$BW_{FN} = 2 \left( \Delta f + B_n \right) = 2 \left( \underline{\underline{\frac{K_f}{\pi}}} + B_n \right)$$

$$B_M = [BW \text{ of } M(t)] = \frac{200\pi}{2\pi} = 100 \text{ Hz} : \text{Highest frequency b/w the sin & cos.}$$

$$300 = 2 \left( \underline{\underline{\frac{K_f}{\pi}}} + 100 \right) \rightarrow \underline{\underline{K_f = 50\pi}}$$

Part D :

$$P_{FM} = \frac{A^2}{2} = 50 \rightarrow \underline{\underline{A = 10 \text{ V (Watt/S)}}}$$

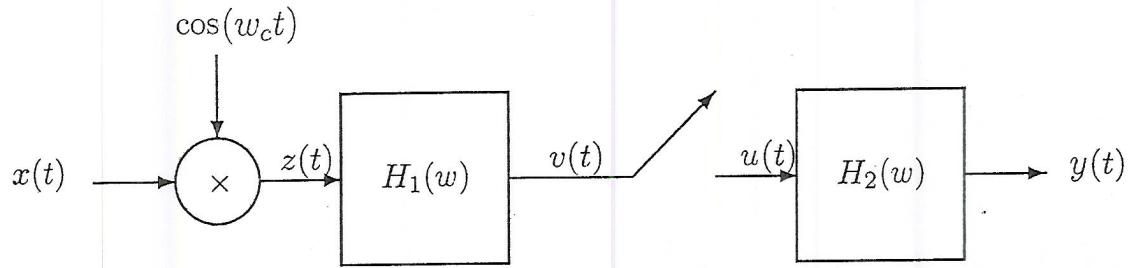
Part E :

$$\begin{aligned} \Phi_{PM} &= A \cos (2\pi f_c t + k_p M(t)) \\ &= \cos (8\pi t + \pi M(t)) = \begin{cases} \cos(8\pi t + 2\pi) = \cos 8\pi t & , 0 \leq t < 1 \\ \cos(8\pi t - \pi) = -\cos 8\pi t & , 1 \leq t < 2 \\ \cos(8\pi t + \pi) = -\cos 8\pi t & , 2 \leq t < 3 \\ \cos(8\pi t + 0) = \cos 8\pi t & , 3 \leq t < 4 \\ \cos(8\pi t - 2\pi) = \cos 8\pi t & , 4 \leq t < 5 \end{cases} \end{aligned}$$

Ambiguity exists. Decoder cannot tell the difference between  $M(t) = 2, 0, -2$  and  $M(t) = 1, -1$ .

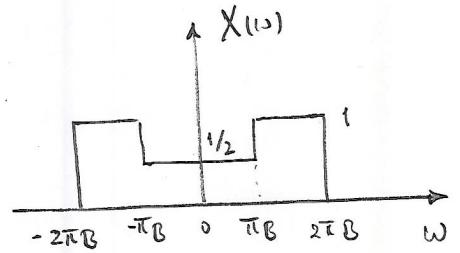
### Problem 3 (30 Points)

Consider the communication system shown below, where the signal  $u(t)$  is obtained by sampling the channel output  $v(t)$  every  $T$  seconds.



Suppose the message signal  $x(t)$  has Fourier transform

$$X(w) = \begin{cases} 0.5 & |w| < \pi B \\ 1 & \pi B \leq |w| < 2\pi B \\ 0 & \text{otherwise} \end{cases}$$



and that the channel has frequency response

$$H_1(w) = H_1(-w) = \begin{cases} 2 & |w - w_c| < \pi B \\ 1 & \pi B \leq |w - w_c| < 2\pi B \\ 0 & \text{otherwise} \end{cases}$$

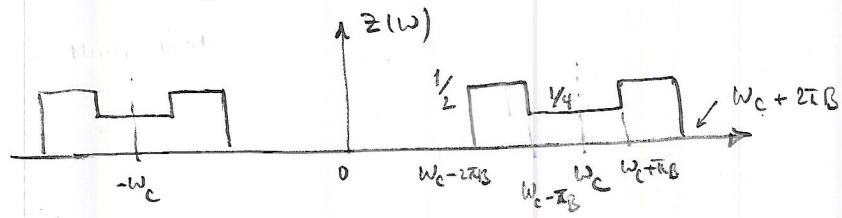
where the carrier frequency  $w_c$  is much bigger than the bandwidth  $2\pi B$  of  $x(t)$ .

**Part A (10 points).** Sketch  $Z(w)$  and  $V(w)$ .

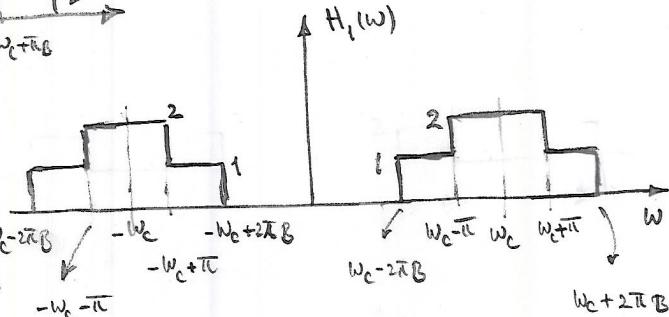
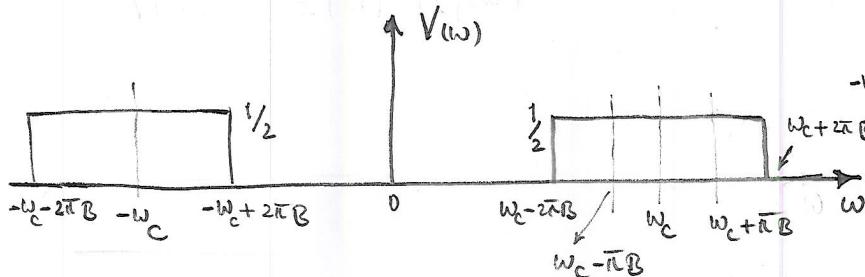
**Part B (10 points).** Sketch  $U(w)$  for  $T = 2\pi/w_c$ .

**Part C (10 points).** Find  $H_2(w)$  so that  $y(t) = x(t)$  for  $T = 2\pi/w_c$ .

Part A :  $Z(w) = \frac{1}{2} X(w + w_c) + \frac{1}{2} X(w - w_c)$

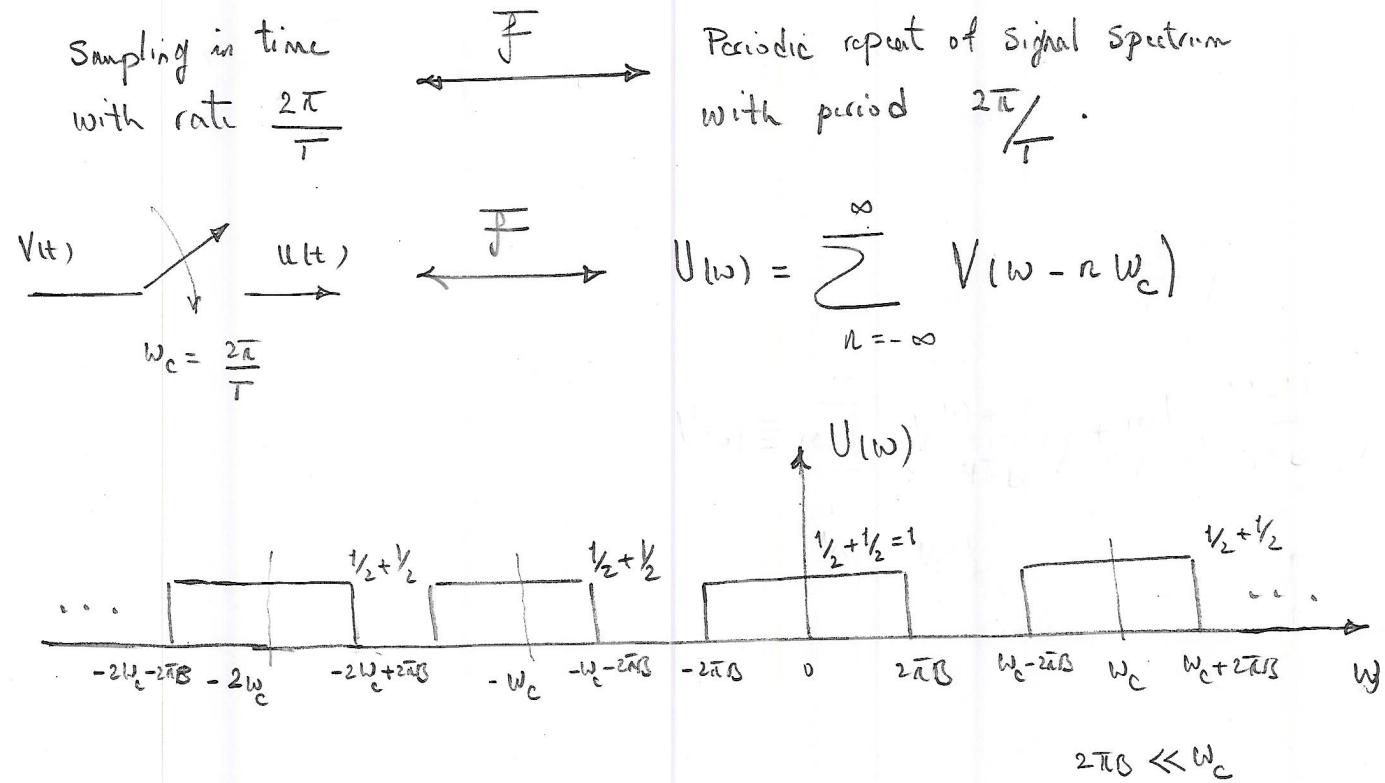


$$V(w) = Z(w) H_1(w)$$



$w_c + 2\pi B$

### Part B :



### Part C :

$$Y(\omega) = U(\omega) H_2(\omega) = X(\omega)$$

