

Colorado State University, Ft. Collins

ECE 421: Telecommunications I  
Fall Semester, 2008

Midterm Exam  
Oct. 28, 2008  
Time: 2 Hours

Name: Solutions

Problem 1 (25 points) \_\_\_\_\_

Problem 2 (55 points) \_\_\_\_\_

Problem 3 (20 points) \_\_\_\_\_

Total: (100) \_\_\_\_\_

### Problem 1 (25 Points)

Let  $x(t) = \text{sinc}(t) \cos(\omega_c t)$ , where  $\omega_c$  is large.

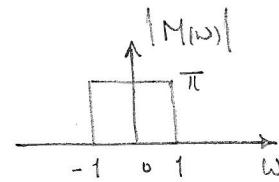
**Part A (10 Points).** Determine (mathematical expression) and sketch the power spectral density  $S_x(\omega)$  of  $x(t)$ .

**Part B (5 Points).** Calculate the power of  $x(t)$ .

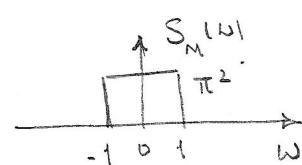
**Part C (10 Points).** Determine the autocorrelation function  $R_x(\tau)$  of  $x(t)$ .

#### Part A

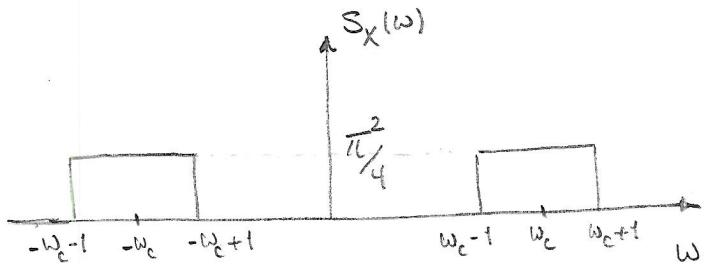
$$m(t) = \text{sinc}(t) \longleftrightarrow M(\omega) = \pi \text{rect}\left(\frac{\omega}{2}\right)$$



$$\begin{aligned} S_M(\omega) &= |M(\omega)|^2 \quad \text{PSD} \\ &= \pi^2 \text{rect}\left(\frac{\omega}{2}\right) \end{aligned}$$



$$\begin{aligned} x(t) = m(t) \cos(\omega_c t) \implies S_X(\omega) &= \frac{1}{4} [S_M(\omega + \omega_c) + S_M(\omega - \omega_c)] \\ &= \frac{\pi^2}{4} \left[ \text{rect}\left(\frac{\omega + \omega_c}{2}\right) + \text{rect}\left(\frac{\omega - \omega_c}{2}\right) \right] \end{aligned}$$



#### Part B

$$P_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega = \frac{1}{2\pi} \left[ 4 \times \frac{\pi^2}{4} \right] = \frac{\pi}{2}$$

#### Part C

$$R_X(\tau) = \mathcal{F}^{-1} \left\{ S_X(\omega) \right\} = \frac{\pi}{2} \text{sinc}(\tau) \cos(\omega_c \tau)$$

## Problem 2 (55 Points)

**Part A (15 Points).** Consider the AM modulated signal  $\phi(t) = (5 + m(t)) \cos(3000t)$  and assume  $m(t) = 2 \cos(200t) + 2.8 \cos(400t)$ . Determine the power efficiency and the modulation index. State whether an envelope detector can be used on this signal.

**Part B (15 Points).** Consider the AM modulated signal  $\phi(t) = \text{rect}(t/2) \cos(20t)$ . Sketch the amplitude spectrum (i.e. magnitude of  $\Phi(w)$ ) and label key frequencies. Estimate the essential bandwidth (in [Hz]) of  $\phi(t)$ .

**Part C (25 Points).** Let  $m(t)$  be a message signal and define the bandpass signals  $x_c(t)$  and  $y_c(t)$  as

$$m(t) = \text{sinc}^2(t)$$

$$x(t) = m(t)\sqrt{2} \cos(2\pi f_c t) - \hat{m}(t)\sqrt{2} \sin(2\pi f_c t)$$

$$y(t) = m(t)\sqrt{2} \cos(2\pi f_c t) + \hat{m}(t)\sqrt{2} \sin(2\pi f_c t)$$

where  $\hat{m}(t)$  is the Hilbert transform of  $m(t)$ . Determine (mathematical expressions) and sketch the amplitude spectra of  $x(t)$  and  $y(t)$ . What is the bandwidth of  $x(t)$ ? What is the power of  $x(t)$ ?

### Part A

- Power efficiency  $\eta = \frac{\frac{1}{2} P_M}{\frac{A^2}{2} + \frac{P_M}{2}} = \frac{5.95}{25 + 5.92} = 19\%$

$A = 5$  dc level.

$$P_M = \frac{1}{2} [P_{M_1} + P_{M_2}] \text{, where } M_1 = 2 \text{ and } M_2 = 2.8.$$

$$= \frac{1}{2} [4 + 2.8^2] \text{. This follows from } \cos(200t) \perp \cos(400t).$$

$$= 5.92$$

- Modulation index  $M = \frac{m_p}{A} = \frac{\max|m(t)|}{A} = \frac{4.8}{5} = 0.96$

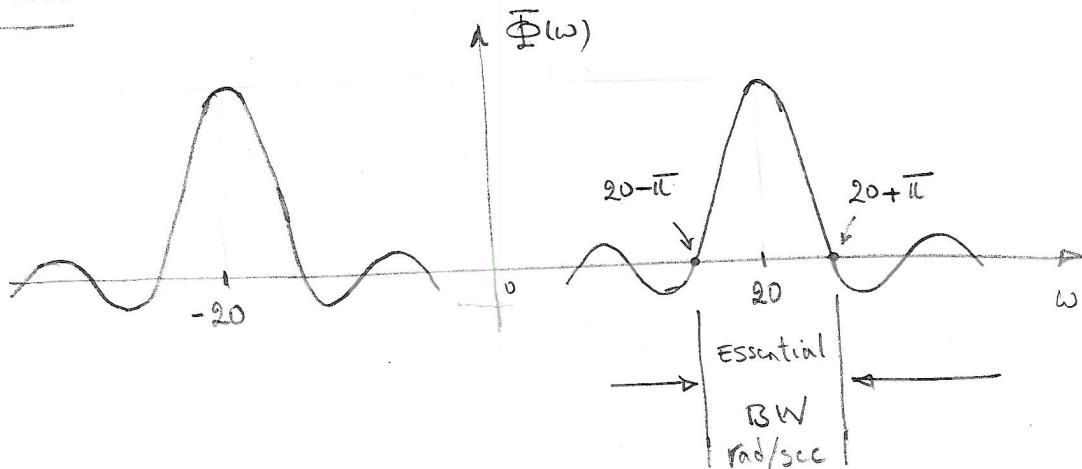
- Yes, envelope detection can be used, since  $5 + M(t) > 0$ , for all  $t$ .

### Part B

$$\text{rect}(t/2) \longleftrightarrow 2 \text{sinc}(\omega)$$

$$\phi(t) = \text{rect}(t/2) \cos(20t) \longleftrightarrow \text{sinc}(\omega+20) + \text{sinc}(\omega-20) = \overline{\Phi}(\omega)$$

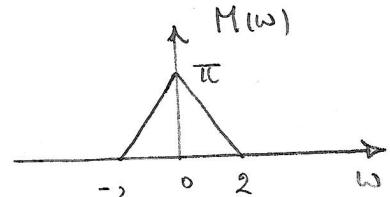
Part B-Cont.



$$\text{Essential BW} = \frac{2\pi}{2\pi} = 1 \text{ Hz}$$

Part C : This is SSB modulation.

$$m(t) = \sin^2(t) \leftrightarrow M(\omega) = \pi \Delta \left( \frac{\omega}{4} \right)$$



$$\hat{M}(t) = \text{HT}\{m(t)\} \leftrightarrow \hat{M}(\omega) = -j \text{sgn}(\omega) M(\omega) \\ = -j \pi \text{sgn}(\omega) \Delta \left( \frac{\omega}{4} \right)$$

$$X(\omega) = \frac{\sqrt{2}}{2} \left[ M(\omega + \omega_c) + M(\omega - \omega_c) + \hat{M}(\omega + \omega_c) - \hat{M}(\omega - \omega_c) \right], \quad \omega_c = 2\pi f_c$$

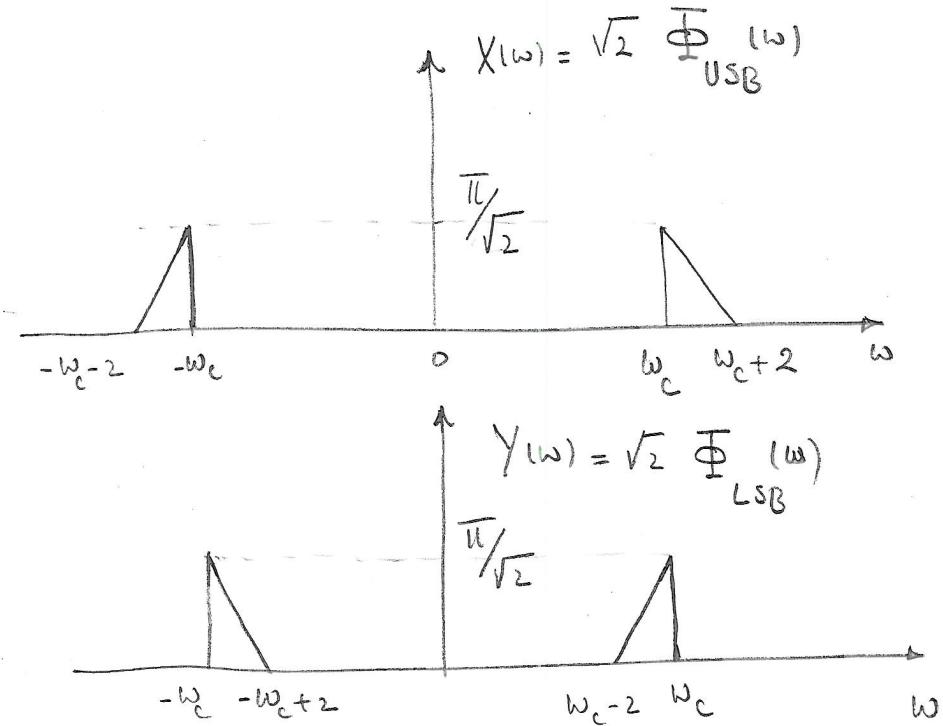
$$= \frac{\sqrt{2}}{2} \left[ M(\omega + \omega_c) + M(\omega - \omega_c) - j \text{sgn}(\omega + \omega_c) M(\omega + \omega_c) + j \text{sgn}(\omega - \omega_c) M(\omega - \omega_c) \right]$$

$$= \frac{\sqrt{2}}{2} \left[ M(\omega + \omega_c) (1 - j \text{sgn}(\omega + \omega_c)) + M(\omega - \omega_c) (1 + j \text{sgn}(\omega - \omega_c)) \right]$$

Similarly

$$Y(\omega) = \frac{\sqrt{2}}{2} \left[ M(\omega + \omega_c) (1 + j \text{sgn}(\omega + \omega_c)) + M(\omega - \omega_c) (1 - j \text{sgn}(\omega - \omega_c)) \right]$$

Part C - Cont: Define  $\phi(t) = m(t) \cos \omega_c t$ . Then



$$BW(X(\omega)) = \underbrace{\omega_c + 2 - \omega_c}_{2 \text{ rad/sec}}$$

$$P_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \left( \left( \frac{\pi^2}{2} \times \frac{2}{2} \right) \times 2 \right) = \frac{\pi}{2}$$

### Problem 3 (20 Points)

Let  $x(t)$  be a *time-limited* signal, such that  $x(t) = 0$  for  $|t| \geq T$ . Show that  $X(w)$  will be completely determined by its sample values  $X(nw_0)$  if  $w_0 \leq \pi/T$ .

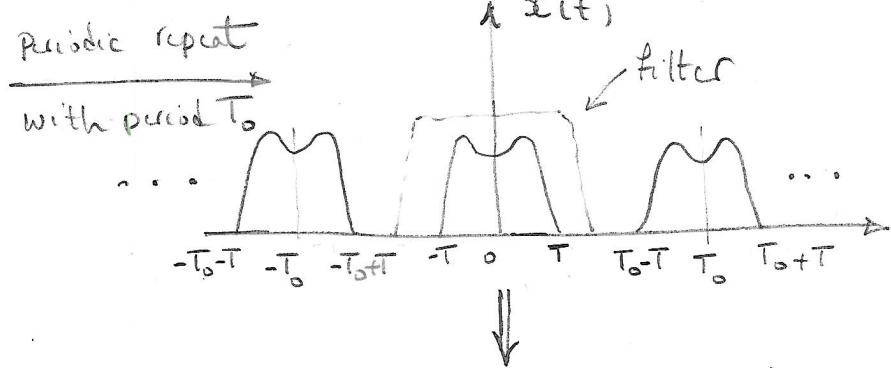
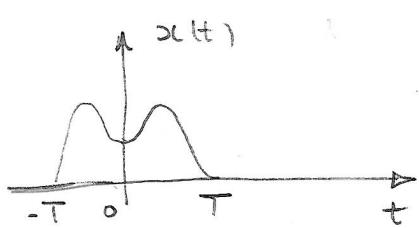
Hint: Use the (exponential) Fourier series for  $\sum_{n=-\infty}^{\infty} \delta(w - nw_0)$ .

Sampling in frequency domain:

$$\begin{aligned}
 X(w) &\xrightarrow[\text{w}_0 \text{ interval}]{\quad} \bar{X}(w) = X(w) \sum_{n=-\infty}^{\infty} \delta(w - nw_0) \\
 &\quad \downarrow \mathcal{F}^{-1} \\
 \bar{x}(t) &= \frac{1}{w_0} x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \quad \text{with } T_0 = \frac{2\pi}{w_0} \\
 &= \frac{1}{w_0} \sum_{n=-\infty}^{\infty} x(t - nT_0)
 \end{aligned}$$

$x(t)$  is time-limited to  $T$ .

For example,



We can reconstruct  $X(w)$  from

$$T_0 - T \geq T$$

$T$ , or

$$w_0 \leq \pi/T \quad \text{Nyquist Condition}$$

this requires

$\bar{x}(t)$  by filtering the center copy of  $x(t) \longleftrightarrow X(w)$  out.