1. (30 Points) Consider the op-amp circuit below.

![Op-Amp Circuit Diagram]

a) What is the Ordinary Differential Equation describing output voltage $y(t)$ with input voltage $x(t)$?

**KCL:** $i_1 = i_2 + i_3$; Substitute device equations:

$$\frac{\dot{y}(t)}{R_1} = -C \frac{d x(t)}{dt} - \frac{y(t)}{R_2}$$

Rearrange:

$$C i \frac{dy(t)}{dt} + \frac{y(t)}{R_2} = -\frac{x(t)}{R_1}$$

or

$$C_1 R_1 R_2 \frac{dy(t)}{dt} + R_1 y(t) = -R_2 x(t)$$

b) What is the Transfer Function, $H(s)$, of this system?

**By inspection:**

$$H(s) = \frac{-R_2}{C_1 R_1 R_2 s + R_1}$$

For alternate:

$$H(s) = \frac{3R_1}{C_1 R_1 R_2 s + 2R_2}$$

$$H(j\omega) = \frac{3}{2 + 2j\omega}$$

$$|H(j\omega)| = \frac{3}{2\sqrt{2}}$$

$$\angle H(j\omega) = -\tan^{-1}(1) = -45^\circ \pm 180^\circ$$

$$y_{ss}(t) = \frac{9}{2\sqrt{2}} \sin(2t + 185^\circ)$$

$$y_{ss}(t) = 3.18 \sin(2t + 185^\circ)$$

$$y(t) = 1.34 \sin(2t - 63.4^\circ)$$

If $R_1 = R_2 = 100K\Omega$, $C_1 = 10\mu F$, and $x(t) = -3\sin(2t)$, what is $y_{ss}(t)$?

Substituting:

$$H(s) = \frac{-100K}{100K s + 100K} = \frac{1}{1 + s}$$

For $s = 2j$, $|H(s)| = \frac{1}{\sqrt{1 + 4}} = \frac{1}{\sqrt{5}} \angle H(s) = -\tan^{-1}(2) \pm 180^\circ = -63.4^\circ$
2. (30 points) Consider the linear time-invariant system with unit step response:
\[ s(t) = (4e^{-4t} - e^{-t})u(t) \]
where \( u(t) \) is the unit step function.

i) What is the system Transfer Function \( H(s) \)? Express this as a rational polynomial.

We know \( u(t) = \frac{d}{dt} g(t) \)  
\[ h(t) = (\frac{-16e^{4t} + e^{-t}}{e^{4t} - e^{-t}})u(t) + (4e^{-4t} - e^{-t}) \frac{du(t)}{dt} \]
Product Rule for Differentiation -  
\[ h(t) = (\frac{-16e^{4t} + e^{-t}}{e^{4t} - e^{-t}})u(t) + (4e^{-4t} - e^{-t})g(t) \]

\[ h(t) = (\frac{-16e^{4t} + e^{-t}}{e^{4t} - e^{-t}})u(t) + 3 \delta(t) \]

\[ H(s) = \frac{1}{s+1} - \frac{16}{s+4} + 3 \]
Inspection

\[ H(s) = \frac{3s^2}{s^2 + 5s + 4} \]

ii) What is the ordinary differential equation (ODE) which describes this system? Use \( x(t) \) to denote the system input and \( y(t) \) to denote the system output.

\[ \frac{d^2 y(t)}{dt^2} + 5 \frac{dy(t)}{dt} + 4y(t) = 3 \frac{d^2 x(t)}{dt^2} \]

iii) Is the system stable?  Y  Why or why not?  
\[ \text{Re} \left( \text{root of CT} \right) = -1, -4 < 0 \]

Does the system have memory?  Y  Why or why not?  \( h(t) \neq 0 \) for \( t \neq 0 \)

Is the system causal?  Y  Why or why not?  \( h(t) = 0 \) for \( t < 0 \)

iv) Another system has a unit step response of similar form:
\[ s_1(t) = 4e^{-4t} - e^{-t} \]

Is this system stable?  N  Why or why not?  Unbounded output as \( t \to -\infty \)

Is this system causal?  N  Why or why not?  \( h(t) \neq 0 \) for \( t < 0 \)
3. (30 Points) Consider the overall system shown below showing output $y(t)$, input $x(t)$, internal signal $z(t)$, and internal subsystems $ODE_1$, $H_2(s)$, and $h_3(t)$.

![Overall System Diagram]

$ODE_1$: $z(t) + 3z(t) = 2x(t)$

$H_2(s) = 1/(s+2)$, and

$h_3(t) = 4\delta(t)$.

a) What are the transfer functions, $H_1(s)$ and $H_3(s)$ for subsystems $ODE_1$ and $h_3(t)$ respectively?

$H_1(s) = \frac{2}{s+3}$, by inspection

$H_3(s) = \int_{-\infty}^{\infty} 4\delta(t)e^{-st}dt = 4\int_{-\infty}^{0} 1\cdot e^{st}dt = 4$

$b) What is the overall system transfer function, $H(s)$ of this system? Express this as a rational polynomial.$

$H(s) = \left(\frac{2}{s+3} + \frac{1}{s+2}\right) \cdot 4$

$= 4 \cdot \frac{2(s+2) + s+3}{(s+3)(s+2)}$

$= \frac{12s + 26}{s^2 + 5s + 6}$

$c) Evaluate the transfer function magnitude $|H(j\omega)|$ for $\omega = 0, 3, and 10 \text{ rad/sec}.$

$|H(0)| = \frac{2.8}{6} = 0.467$

$|H(3j)| = \sqrt{360^2 + 28^2} \approx \sqrt{20250} \approx 299.4$

$|H(10j)| = \sqrt{(1200^2 + 28^2) \cdot \sqrt{1536^2 + 58^2}} = \sqrt{1536^2 + 58^2} = 111.8$

$d) Is this overall system Causal? \checkmark \hspace{1cm} BIBO stable? \checkmark \hspace{1cm} Time Invariant? \checkmark \hspace{1cm} Linear? \checkmark$

$e) (Extra credit) What kind of filter would this overall system represent? Low-Pass \hspace{1cm} H(j\omega) decreases with frequency$
4. (10 Points) Given \( x(t) = 5u(t + 2) - 2u(t) + 3u(t - 2) - 6u(t - 4) \), find and plot the odd part of this function, \( x_0(t) \).