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Key Points:

- A new formulation for the turbulent Prandtl number is proposed
- Performance of the proposed Pr_t is evaluated to highlight its usefulness
- Our findings will be useful for modeling geophysical flows

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A simple turbulence model for stably stratified wall-bounded flows

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Abstract In this study, we present a simple zero-equation (algebraic) turbulence closure scheme as well as the standard k-ϵ model for stably stratified wall-bounded flows. We do this by proposing a parameterization for the turbulent Prandtl number (Pr_t) for stably stratified flows under the influence of a smooth solid wall. The turbulent Prandtl number is the linking bridge between the turbulent momentum and scalar fluxes in the context of Reynolds-averaged Navier-Stokes (RANS) simulations. Therefore, it is important to use appropriate parameterizations for Pr_t in order to define the right level of momentum and scalar mixing in stably stratified flows. To date, most of the widely used parameterizations for Pr_t in stably stratified flows are based on data obtained from homogeneous shear flows experiments and/or direct numerical simulations (i.e., statistics are invariant under translations) and are usually formulated as functions of the gradient Richardson number (Ri_g). The effect of the wall boundary is completely neglected. We introduce a modified parameterization for Pr_t that takes into account the inhomogeneity caused by the wall coupled with the effects of density stratification. We evaluate the performance of the modified Pr_t by using a zero-equation turbulence model for the turbulent viscosity that was proposed by Munk and Anderson (1948) as well as the standard k-ϵ model to simulate a one-dimensional stably stratified channel flow. Comparison of the one-dimensional simulation results with direct numerical simulation (DNS) of stably stratified channel flow results show remarkable agreement.

1. Introduction

The majority of geophysical flows (e.g., in the oceans, lakes, estuaries, and the atmosphere) are substantially influenced by stable density stratification. The density stratification causes buoyancy forces that can significantly influence the mixing of both momentum and scalars [*Rodi*, 1987]. Hence, it is important to develop turbulence models with the ability to predict mixing in such flows. However, many geophysical flows are also influenced by wall (solid) boundaries such as in the coastal ocean, lakes, and the atmospheric boundary layer (ABL). The presence of the wall introduces inhomogeneities in the flow which causes complex turbulent structures close to the wall (i.e., in the so-called inner wall region, see for example, *Pope* [2000] for more details). This is in contrast to free-shear flows which are able to develop without the confining influence of the wall. Hence, it is expected that the mixing dynamics in a stably stratified wall-bounded flow should be different than the simpler homogeneous stably stratified flow, where the fluctuating quantities (such as the velocity and density fluctuations) are statistically homogeneous. Statistical homogeneity implies that the statistics are invariant under translation [*Pope*, 2000].

Numerical simulations of turbulent flows range from direct numerical simulation (DNS) where the highly nonlinear Navier-Stokes equations together with the continuity and density transport equations are solved directly to yield the instantaneous flow fields without recourse to a turbulence model; to the Reynolds-averaged Navier-Stokes (RANS) simulation where an averaged (statistical) flow solution is obtained. DNS is one of the commonly used simulation techniques to gain better fundamental understanding of turbulence. On the other hand, RANS simulations are used for practical applications where the emphasis is on better understanding of complex phenomena such as bottom boundary layer mixing in the coastal ocean and internal wave-driven mixing in the ocean [*Burchard*, 2002]. It is important to note that DNS is prohibitively expensive for most practical flow problems and therefore only suitable to very idealized flows [*Pope*, 2000]. However, there have been a number of DNS and large-eddy simulation (LES) studies that have focused on stably stratified wall-bounded flows [see e.g., *Armenio and Sarkar*, 2002; *Nieuwstadt*, 2005; *García-Villalba and del Álamo*, 2011]. In the RANS formulation, the averaging process (using Reynolds decomposition) results in additional terms known as the Reynolds stresses (turbulent momentum fluxes) and turbulent scalar fluxes in the mean momentum and scalar transport equations, respectively. These extra terms imply that the number of unknowns is greater than the number of available (mean flow) equations leading to an undetermined system commonly referred to as the turbulence closure problem.

A common and widely employed approach to close the RANS equations is through the use of the turbulent-viscosity hypothesis together with the gradient-diffusion hypothesis. The basic assumption in these hypotheses is that the turbulent momentum flux (or turbulent scalar flux) is transported down (i.e., aligned with) the mean gradient of the respective averaged flow variable. Essentially, using these hypotheses, the turbulence closure problem reduces to the prediction of an eddy (turbulent) viscosity (ν_t) and a corresponding eddy (turbulent) diffusivity (κ_t). Although the assumptions in these hypotheses are seldom valid in many flows, they are nevertheless widely used to close the RANS equations due to their simplicity. There are numerous approaches for modeling ν_t and κ_t ranging from simple zero-equation (algebraic) models to more sophisticated two-equation turbulence models (e.g., the k-ϵ model by *Jones and Launder* [1972]). For stably stratified flows, parameterizing both ν_t and κ_t in such a way as to yield the right levels of both momentum and scalar (density) mixing has proved to be challenging. Therefore, it is essential to develop robust RANS models that have the benefit of fast calculation speeds along with accuracy. This continues to remain an open research problem for the turbulence modeling community.

Many RANS models use the turbulent Prandtl (or Schmidt) number $(Pr_t = \nu_t/\kappa_t)$ to link the turbulent momentum and scalar fluxes [*Venayagamoorthy and Stretch*, 2010]. In a uni-directional planar shear flow (e.g., a turbulent channel flow), ν_t is defined as

$$\nu_t = \frac{-\overline{u'w'}}{d\bar{U}/dz}, \tag{1}$$

and κ_t is given by

$$\kappa_t = \frac{-\overline{\rho'w'}}{d\bar{\rho}/dz}, \tag{2}$$

where $\overline{u'w'}$ is the Reynolds stress (turbulent momentum flux), $\overline{\rho'w'}$ is the turbulent density flux, z is the normal distance from the wall, $d\bar{U}/dz$ is the mean shear rate (often denoted as S), and $d\bar{\rho}/dz$ is the vertical mean density gradient. It is easy to see from the definition of the turbulent Prandtl number, the importance of parameterizing it to correctly capture the turbulent momentum and scalar fluxes. This is even more important when the scalar is active such as in stably stratified flows. There are a number of parameterizations for Pr_t for stably stratified flows such as those proposed many decades ago by *Munk and Anderson* [1948] (hereafter *MA*) and more recently by *Venayagamoorthy and Stretch* [2010] (hereafter *VS*), to mention a couple. The two main factors that tend to influence the turbulent mixing in stably stratified wall-bounded flows are the density stratification and the solid boundary (*MA*). However, in many flow conditions such as in the mixed layer, the effect of the boundary is very limited and can thus be neglected. Based on such arguments, most turbulent Prandtl number parameterizations only consider the effect of the density stratification and hence by default are only applicable to homogeneous (free) shear flows. But in a wall-bounded flow, the presence of the solid wall introduces inhomogeneities in the flow that causes anomalous transport of momentum and scalar close to the wall [*Launder and Spalding*, 1972; *Crimaldi et al*., 2006].

In this paper, we discuss the behavior of the turbulent Prandtl number in the presence of a solid wall and introduce a new parameterization for Pr_t that accounts for the presence of the wall along with stratification. In order to test the new parameterization for Pr_t, a zero-equation (algebraic) RANS model that makes use of the modified turbulent viscosity (ν_t) proposed by *MA* as well as the two-equation standard k-ϵ turbulence closure scheme, are developed in MATLAB and the results are compared with the DNS data of stably stratified channel flow of *García-Villalba and del Álamo* [2011]. We also compare the Pr_t parameterization of *MA* for homogeneous flows to highlight the shortcomings of homogeneous formulation for Pr_t in predicting both momentum and scalar mixing correctly in wall-bounded flows. In what follows, we present the parameterization of the turbulent Prandtl number in section 2, followed by a description of the numerical simulation setup in section 3 and results in section 4. We provide a summary in section 5.

2. Parameterization of the Turbulent Prandtl Number

It has been shown [e.g., *Schumann and Gerz*, 1995, *VS*] that in a stably stratified homogeneous flow, Pr_t can be defined as

$$Pr_t = \frac{Ri_g}{R_f} + Pr_{t0}, \tag{3}$$

where Pr_{t0} is the neutral Prandtl number in the limit of zero stratification in a homogeneous shear flow. Pr_{t0} has been shown to be close to unity [*Kays et al*., 1993; *Kays*, 1994]. In equation (3), Ri_g is the gradient Richardson number and is defined as

$$Ri_g = \frac{N^2}{S^2}, \tag{4}$$

where $N = \sqrt{-(g/\rho_0)(d\bar{\rho}/dz)}$ is the Brunt-Väisälä or buoyancy frequency, g is the gravitational acceleration, and ρ_0 is the background density of the fluid. N represents the frequency of a fluid particle oscillating in a flow when displaced from its stable position and provides a measure of the strength of the density stratification. R_f is the flux Richardson number which in shear flows is conventionally defined as [*Peltier and Caulfield*, 2003]

$$R_f = \frac{-B}{P}, \tag{5}$$

where $B = -g/\rho_0(\overline{\rho' w'})$ is the buoyancy flux and $P = -\overline{u'w'}(d\bar{U}/dz)$ is the rate of production of the turbulent kinetic energy ($k = 0.5u_i{}'2$, using the Einstein summation convention). Direct measurement of the buoyancy flux in stratified flows is not trivial due to technical difficulties as well as complex physical processes such as contamination from internal waves. To circumvent this problem, indirect approaches are used to infer B from the scalar dissipation rate (χ), the dissipation rate of the turbulent kinetic energy (ϵ), N, and the Thorpe overturning length scale (L_T). *Osborn and Cox* [1972] have defined the buoyancy flux indirectly by assuming that the advective terms are negligible. To date, there is no general consensus on a universal parameterization for R_f even though a number of parameterizations exist for R_f. This is mainly due to the lack of evidence on what the behavior of R_f should be under very strong stable stratification in high-Reynolds number flows. Laboratory experiments and direct numerical simulations remain inconclusive about this issue due to Reynolds number limitations. Field experiments tend to show quite a bit of scatter. However, recent DNS studies, in particular those of *Mashayek and Peltier* [2013] and *Mashayek et al*. [2013], have sought to increase the Reynolds number limit of numerical simulations to realistic geophysical flows. These recent results indicate that R_f might be highly variable and difficult to parameterize in free-shear layers, especially at strong stratification. Regardless, there have been some formulations that have gained acceptance. For example, *Osborn* [1980] has proposed that $R_f \leq 0.17$ based on a few laboratory experiments of shear flows. It should be noted, however, that there is growing evidence that the assumptions of fully developed turbulence, stationarity, and homogeneity, inherent in Osborn's formulation as well as a constant value of R_f are highly debatable [see e.g., *Mashayek and Peltier*, 2013; *Mashayek et al*., 2013; *Ivey et al*., 2008]. *Mellor and Yamada* [1982] (hereafter *MY*) have proposed a parameterization where $R_f \leq 0.25$ given by

$$R_f = 0.725\left[Ri_g + 0.186 - \left(Ri_g^2 - 0.316Ri_g + 0.0346\right)^{1/2}\right]. \tag{6}$$

Recently, *Canuto et al*. [2001] using the data set of *Maderich et al*. [1995] have shown that the flux Richardson number is directly related to Ri_g. They show that R_f increases with Ri_g from zero for neutral (zero) stratification and asymptotes to a value $R_{f\infty} \approx 0.25$ around $Ri_g \approx 1$. The measured flux Richardson numbers show an exponential behavior as a function of the gradient Richardson number. Here we propose a simple exponential relationship for R_f as a function of Ri_g

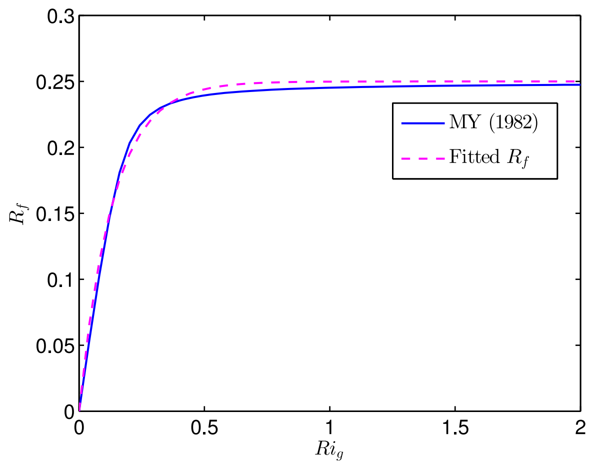

Figure 1. The flux Richardson number (R_f) as a function of the gradient Richardson number (Ri_g).

$$R_f = R_{f\infty}\left[1-exp\left(-\gamma Ri_g\right)\right], \quad (7)$$

where $R_{f\infty}=0.25$ and γ is a constant that is set equal to 7.5. The behavior of R_f as a function of Ri_g given by equations (6) and (7) are shown in Figure 1. It is clear that the proposed formulation is very similar to the formulation of *MY* given in equation (6).

As shown in equation (3), the homogeneous Pr_t increases linearly without bound with the gradient Richardson number (Ri_g). Some studies have been carried out to highlight the effect of the wall on Pr_t such as *Kawamura et al*. [1998], *Crimaldi et al*. [2006], and *Srinivasan and Papavassiliou* [2010]. *Crimaldi et al*. [2006] have measured Pr_t in an unstratified boundary layer flow in the laboratory. They have observed that Pr_t is much higher than unity near the bed (i.e., for $z^+ = zu_\tau/\nu < 30$, where z^+ is the wall unit, u_τ is the friction or turbulent velocity, and ν is the kinematic or molecular viscosity) and decreases almost linearly to about unity in the free-stream. This implies that at the wall Pr_t is a maximum and decreases to the value of the homogeneous shear flow case in the outer flow region. Coincidentally, this behavior is in agreement with the linear shear stress distribution in a channel flow given by

$$\tau = \tau_w\left(1-\frac{z}{D}\right), \quad (8)$$

where τ is the shear stress at a level z from the wall, τ_w is the (maximum) shear stress at the wall, D is the flow depth, and z is bounded in the range $0 \le z \le D$. We note that the linear shear stress distribution given in equation (8) holds strictly only for channel and pipe flows. However, the shear stress distribution from DNS of boundary-layer flow [*Jiménez et al*., 2010] indicates that assuming a linear distribution is reasonable in the logarithmic region. *Kawamura et al*. [1998] have performed a DNS study of heat transfer in a channel flow. Their results show that beginning in the log-law region where $z^+ \approx 30$, Pr_t starts to decrease to about unity in the outer free-stream. *Srinivasan and Papavassiliou* [2010] show that for an unstratified wall-bounded flow for fluids with molecular Prandtl number of $Pr = \nu/\kappa_m > 0.7$ with κ_m defined as the molecular diffusivity, the neutral turbulent Prandtl number starts from values above unity at the wall and decreases to almost the molecular Prandtl number (Pr) at the free-surface. It is also worth noting in passing that *Launder and Spalding* [1972] indicate that Pr_t ought to follow a linear distribution with a higher value at the wall that decreases as the free-stream is approached. Therefore, the following formulation for the unstratified (neutral) Pr_t in a wall-bounded flow can be proposed

$$Pr_{tw0} = \left(1-\frac{z}{D}\right)Pr_{twd0} + Pr_{t0}, \quad (9)$$

where Pr_{twd0} is the difference between the neutral turbulent Prandtl number at the wall (Pr_{tw0}) and the neutral turbulent Prandtl number for a homogeneous shear flow (Pr_{t0}). DNS data in the near-wall region $(z^+ \approx 30)$ indicate that Pr_{tw0} varies in the range 1–1.5 for different molecular Prandtl numbers (Pr) (e.g., see the discussion in *McEligot and Taylor* [1996]). *Crimaldi et al.* [2006] show that Pr_{tw0} is around 2 in the log-law region and tapers off almost linearly to close to unity as the free-stream is approached. Most RANS simulations make use of the logarithmic law of the wall to model wall-bounded flows, therefore here, we use $Pr_{tw0} \approx 1.1$. Different values for the neutral turbulent Prandtl number (Pr_{t0}) in homogeneous shear flows have been suggested but there is consensus that its value is very close to unity [*VS*]. However, *VS* have argued that $Pr_{t0}=0.7$ using DNS data of homogeneous shear flows, in agreement with several other studies [e.g., *Schumann and Gerz*, 1995]. We use this value for the purpose of this study. For the unstratified case, equation (9) provides a linear correction to Pr_{t0}, with the value at the beginning of the logarithmic region equal to Pr_{tw0}. This linear formulation is in agreement with the observed DNS data, experiment of *Crimaldi et al.* [2006] and also proposition of *Launder and Spalding* [1972]. It is also worth noting that if the wall effect is removed, the formulation given by equation (9) reverts back to neutral value in homogeneous unstratified shear flows.

Let us now consider how to extend this discussion to stably stratified wall-bounded flows. Using equation (8) and evoking the turbulent-viscosity hypothesis, it is straightforward to show that the turbulent viscosity (for the log-law region) is given by

$$\nu_t = \frac{u_\tau^2}{S}\left(1-\frac{z}{D}\right), \tag{10}$$

where $u_\tau = (\tau_w/\rho)^{1/2}$ is the friction velocity. Furthermore, using the formulation by *Osborn* [1980], the turbulent diffusivity (κ_t) is commonly assumed to be given by

$$\kappa_t = \Gamma \frac{\epsilon}{N^2}, \tag{11}$$

where ϵ is the dissipation rate of the turbulent kinetic energy (k) and Γ is the mixing efficiency and is related to R_f as

$$\Gamma = \frac{R_f}{1-R_f}. \tag{12}$$

It is worth noting here that Osborn's formulation for the turbulent diffusivity assumes stationarity and has been shown to be an oversimplification of the mixing problem by *Smyth et al.* [2001] and a number of other more recent studies [e.g., *Mashayek et al.*, 2013]. However, it is a widely used formulation for quantifying mixing in oceanic flows. With this caveat in mind, we retain this formulation and proceed by dividing equation (10) by equation (11) to get an expression for the stratified component of the turbulent Prandtl number (Pr_t) as follows

$$Pr_t = \frac{u_\tau^2/S\left(1-\frac{z}{D}\right)}{\Gamma(\epsilon/N^2)}. \tag{13}$$

Durbin and Pettersson Reif [2011] discuss that in the constant-stress (log-law) region, the equilibrium assumption holds between the production rate of the turbulent kinetic energy (P) and the dissipation rate of the turbulent kinetic energy (ϵ), i.e., $P=\epsilon$. This implies that for unstratified channel flows, ϵ can be expressed as

$$\epsilon = \frac{u_\tau^3}{\kappa z}, \tag{14}$$

where κ is the von Kármán constant (assumed to be ≈ 0.40 in this study). Similarly, evoking the equilibrium assumption between the production rate of the turbulent kinetic energy (P), the dissipation rate of the turbulent kinetic energy (ϵ), and the buoyancy flux (B) in the logarithmic region of a stably stratified wall-bounded flow yields

$$P+B=P(1-R_f)=\epsilon, \tag{15}$$

which can be rearranged to get the dissipation rate (ϵ) as

$$\epsilon = (1-R_f)\frac{u_\tau^3}{\kappa z}. \tag{16}$$

Substituting equation (16) into equation (13) gives Pr_t as

$$Pr_t=\frac{N^2}{R_f}\left(\frac{\kappa z}{u_\tau}\right)\frac{1}{S}\left(1-\frac{z}{D}\right). \tag{17}$$

From the classical log-law, the mean velocity gradient is given by

$$S=\frac{\partial \bar{U}}{\partial z}=\frac{u_\tau}{\kappa z}. \tag{18}$$

This can now be substituted into equation (17) to get

$$Pr_t=\frac{N^2}{S^2}\frac{1}{R_f}\left(1-\frac{z}{D}\right)=\frac{Ri_g}{R_f}\left(1-\frac{z}{D}\right). \tag{19}$$

Analogous to homogeneous Pr_t (equation (3)) which is a combination (algebraic sum) of the stratified Pr_t (i.e., Ri_g/R_f) and neutral turbulent Prandtl number (Pr_{t0}) in conjunction with the discussion above, we propose a Pr_t for stably stratified wall-bounded flows, by combining the wall-bounded stratified Pr_t (equation (19)) and unstratified wall-bounded Pr_t (i.e., Pr_{tw0} in equation (9)) as follows

$$Pr_t=\left(1-\frac{z}{D}\right)\frac{Ri_g}{R_f}+\left(1-\frac{z}{D}\right)Pr_{twd0}+Pr_{t0}. \tag{20}$$

In equation (20), the effect of the buoyancy is considered through Ri_g/R_f term and the wall effect is taken into account by using $(1-z/D)$. In this paper, we show that it is important to take this behavior of Pr_t into account when modeling stably stratified wall-bounded flows in order to simultaneously predict the mean velocity and density profiles correctly.

With any given parameterization for Pr_t, we need to make use of either ν_t or κ_t to calculate one from the other. It is common for the turbulent viscosity (ν_t) to be parameterized. Hence, accurate parameterization of the turbulent viscosity (ν_t) in a RANS model is very important. For an unstratified fully developed turbulent channel flow with a hydrostatic pressure distribution and a logarithmic velocity profile, it can be mathematically shown that the turbulent viscosity (ν_t) is simply a parabolic function of depth given by [*Rodi*, 1993]

$$\nu_{t0}=\kappa u_\tau z(1-z/D). \tag{21}$$

This equation only takes into account the effect of boundaries in generating the turbulence and is appropriate for a zero-equation RANS model. Equation (21) is the turbulent viscosity (ν_t) for a neutrally stable flow and thus does not incorporate the effect of density stratification. *MA* argue that as $Ri_g \to 0, \nu_t \to \nu_{t0}$ and when $Ri_g \to \infty, \nu_t \to 0$. Based on these arguments, they provide a simple modification to account for density stratification as a function of Ri_g. Here we adopt their formulation to modify the parabolic turbulent viscosity (ν_t) to equation (22) for a stably stratified flow as a function of Ri_g given by

$$\nu_t=\kappa u_\tau z(1-z/D)\left(1+\beta Ri_g\right)^\alpha, \tag{22}$$

where β and α are empirical constants with values of 10 and $-1/2$, respectively (*MA*).

In more sophisticated turbulence closure models such as the standard k-ϵ model, additional transport equations are solved to evaluate ν_t. In the standard k-ϵ model, the turbulent viscosity is given by

$$\nu_t=(1-R_f)C_\mu\frac{k^2}{\epsilon}, \tag{23}$$

where $C_\mu=(|\overline{u'w'}|/k)^2$ is the turbulent viscosity parameter and is usually considered to be roughly 0.09 [*Karimpour and Venayagamoorthy*, 2013]. The transport equation of k for a stratified inhomogeneous flow in the standard k-ϵ closure scheme is given by

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$$\frac{\partial k}{\partial t}+\overline{U_j}\frac{\partial k}{\partial x_j}=P-\epsilon+B+\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right), \tag{24}$$

where the last term is the transport of k due to existence of solid wall which is modeled using the gradient-diffusion hypothesis with σ_k as the turbulent Prandtl number for k. Furthermore, in the standard k-ϵ model, the dissipation rate of the turbulent kinetic energy (ϵ) is obtained through an empirical transport equation. This equation is a dimensionally consistent analogy to the transport equation of k [*Durbin and Pettersson Reif*, 2011] and for high-Reynolds number flows is given by

$$\frac{\partial \epsilon}{\partial t}+\overline{U_j}\frac{\partial \epsilon}{\partial x_j}=C_{\epsilon 1}P\frac{\epsilon}{k}-C_{\epsilon 2}\epsilon\frac{\epsilon}{k}+C_{\epsilon 3}B\frac{\epsilon}{k}+\frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right). \tag{25}$$

In equation (25), $C_{\epsilon 1}$, $C_{\epsilon 2}$, and $C_{\epsilon 3}$ are empirical constants for the "production of dissipation," "dissipation of dissipation," and "buoyancy flux of dissipation," respectively. Also, σ_ϵ is an empirical dissipation turbulent Prandtl number. All of the constants in the k-ϵ model except for $C_{\epsilon 3}$ have standard values as follows: $C_{\epsilon 1}=1.44$; $C_{\epsilon 2}=1.92$; $\sigma_k=1.0$; and $\sigma_\epsilon=1.3$. In contrast to the other constants that have more or less universal values as shown above, there is still no consensus on the value of the buoyancy parameter $(C_{\epsilon 3})$. However, *Rodi* [1987] has shown that $C_{\epsilon 3}\approx 0$ is a reasonable value based on some successful numerical simulations of stratified flows. Following his work, we have also used $C_{\epsilon 3}=0$ in our k-ϵ based RANS simulations.

In this study, we employ the *MA* formulation for the turbulent viscosity (ν_t) as well as the standard k-ϵ closure scheme and use the proposed formulation for Pr_t in equation (20) to highlight the importance of considering the effect of the wall in modeling the scalar transport. We do this by comparing numerical simulations results for a one-dimensional fully developed channel flow case with results from three-dimensional DNS of channel flow of *García-Villalba and del Álamo* [2011].

3. Numerical Model

A one-dimensional fully developed smooth-wall channel flow similar to DNS channel flow of *García-Villalba and del Álamo* [2011] at a friction Reynolds number of $Re_\tau=u_\tau D/\nu=550$ is simulated in this study using both turbulent viscosity (ν_t) proposed by *MA* as well as the standard k-ϵ model. Two different stratifications with friction Richardson numbers of $Ri_\tau=\Delta\rho gD/\rho_0 u_\tau^2=60$ and 120 were used, where $\Delta\rho$ is the density difference between the top and bottom of the channels. The RANS results are compared with the DNS data of *García-Villalba and del Álamo* [2011]. To our knowledge, this is the only available highly resolved DNS database of stably stratified turbulent channel flow.

We do this by solving the 1-D RANS momentum and scalar transport equations. The 1-D RANS momentum equation with the Boussinesq approximation is given by

$$\frac{\partial \bar{U}}{\partial t}=-\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}+\nu\frac{\partial^2\bar{U}}{\partial z^2}-\frac{\partial}{\partial z}\left(\overline{u'w'}\right), \tag{26}$$

which using the turbulent-viscosity hypothesis can be rewritten as

$$\frac{\partial \bar{U}}{\partial t}=-\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}+\nu\frac{\partial^2\bar{U}}{\partial z^2}+\frac{\partial}{\partial z}\left(\nu_t\frac{\partial \bar{U}}{\partial z}\right). \tag{27}$$

For simplicity, a pressure-driven flow (i.e., constant pressure gradient) is assumed so that the pressure can be decoupled from the velocity. Using the hydrostatic pressure distribution, the pressure term is then given by

$$-\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x}=-\frac{1}{\rho_0}\frac{\partial(\rho_0 gh)}{\partial x}=-g\frac{\partial h}{\partial x}=-gS, \tag{28}$$

where $S=-u_\tau^2/gD$ is the slope of the free-stream. The 1-D RANS scalar transport equation is given by

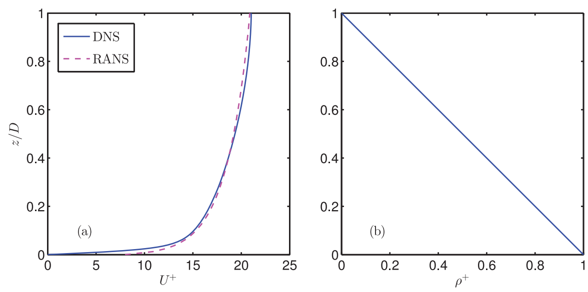

Figure 2. (a) Fully developed unstratified velocity profile and (b) initial density profile.

$$\frac{\partial \bar{\rho}}{\partial t} = \kappa_m \frac{\partial^2 \bar{\rho}}{\partial z^2} - \frac{\partial}{\partial z}\left(\overline{\rho' w'}\right), \tag{29}$$

and can be simplified using the gradient-diffusion hypothesis as

$$\frac{\partial \bar{\rho}}{\partial z} = \kappa_m \frac{\partial^2 \bar{\rho}}{\partial z^2} + \frac{\partial}{\partial z}\left(\kappa_t \frac{\partial \bar{\rho}}{\partial z}\right). \tag{30}$$

The above RANS equations cannot be solved analytically and require a numerical solution methodology. To perform a numerical simulation, the governing equations have to be discretized in space and in time in order to convert the partial differential equations into a set of algebraic equations. In this study, a second-order accurate central difference scheme using the finite volume method is employed for spatial discretization. The temporal terms are discretized using a semi-implicit θ-method, where θ is the implicitness parameter that can range from 0 to 1. The θ-method can be represented as

$$\frac{\partial T}{\partial t} = \theta f\left(T^{n+1}\right) + (1-\theta) f(T^n), \tag{31}$$

where T is an arbitrary variable (e.g., $\bar{U}$), dependent on time and space and f shows a spatial function. The θ-method improves the stability and/or accuracy of the method through the weighting of the explicit and implicit terms, using the implicitness parameter θ [*Casulli and Cattani*, 1994]. For example, when θ=0 the method is first-order accurate and fully explicit while for θ=1 the method is fully implicit with first-order accuracy. When θ=0.5, the scheme is usually known as Crank-Nicolson method [*Moin*, 2010], and is a semi-implicit, second-order accurate scheme that evenly distributes the weighting of the explicit and implicit terms. The channel flow governing equations are stable for $0.5 \le \theta \le 1$. For this study, $\theta \approx 0.7$ is used since it yielded stable solutions with no oscillations.

A no-slip boundary condition at the solid wall (i.e., $\bar{U}$=0) and free-slip boundary condition at the free-surface (i.e., $\partial \bar{U}/\partial z$=0) are imposed. The no-slip boundary condition requires modeling the very thin boundary layer region (the so-called near-wall region where $z^+ < 30$). A commonly used technique is to apply the velocity boundary condition at some distance away from the wall, where the logarithmic velocity profile begins (i.e., $z^+ > 30$). Assuming the existence of a logarithmic velocity profile and using a linear shear stress distribution, the solid wall boundary condition can be represented as

$$\frac{\partial \overline{U_1}}{\partial z} = \frac{C_D}{\nu_t} |\overline{U_1}| \overline{U_1}, \tag{32}$$

where $\overline{U_1}$ is the velocity at the first grid point of the flow domain and C_D can be interpreted as a drag coefficient as

$$C_D = \left[\frac{1}{\kappa} \ln\left(\frac{z_1}{z_0}\right)\right]^{-2}, \tag{33}$$

where z_0 is the roughness height such that the logarithmic velocity profile goes to zero, while z_1 is the physical distance of the first grid point from the wall, located in the log-law region.

Similarly, the standard k-ϵ model is incapable of modeling the near-wall region due to excessive overprediction of the turbulent viscosity (ν_t) in this intricate near-wall region [*Karimpour and Venayagamoorthy*, 2013]. In the standard k-ϵ closure scheme, modeling the near-wall region is avoided by employing wall-functions. The wall-functions impose the boundary conditions at some distance away from the wall in the log-law

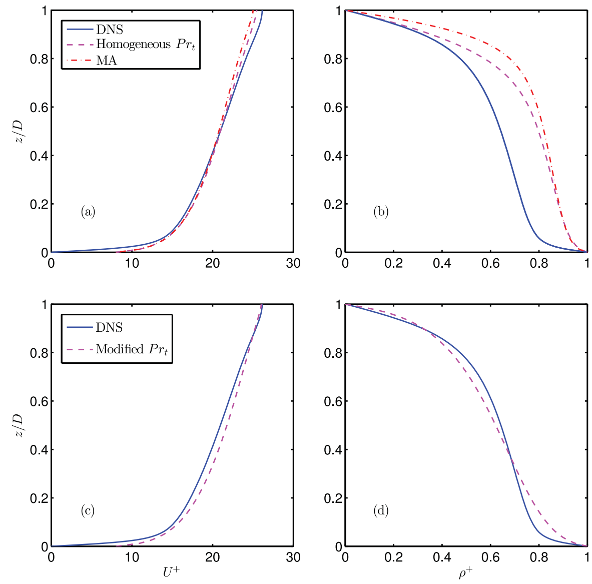

Figure 3. (left) Velocity profiles and (right) density profiles for $Ri_\tau=60$ obtained from the zero-equation closure scheme using (a, b) the Pr_t formulations given by equations (3) and (37) and (c, d) the modified Pr_t formulation (equation (20)) compared with the channel flow DNS data of *García-Villalba and del Álamo* [2011].

region. Assuming the existence of a logarithmic velocity profile, the boundary condition for the mean velocity at the first grid point is given by

$$\overline{U_1}=\frac{u_\tau}{\kappa}\ln\left(\frac{z_1}{z_0}\right). \tag{34}$$

Consequently, the boundary conditions for the turbulent kinetic energy (k) and ϵ (i.e., at the first point (z_1)) are given by

$$k_1=\frac{u_\tau^2}{\sqrt{C_\mu}}, \tag{35}$$

and

$$\epsilon_1=\frac{u_\tau^3}{\kappa z_1}. \tag{36}$$

The natural boundary condition for the density would be a Neumann boundary condition at both or at least one of the boundaries (i.e., $\partial\bar{\rho}/\partial z=0$). However, applying a Neumann boundary condition results in a fully mixed density field $(\partial\bar{\rho}/\partial z=0)$ across the whole water column. In order to be able to evaluate the effect of the stable stratification and also the efficacy of the proposed Pr_t's, the density profile is kept constant at both boundaries (i.e., Dirichlet boundary conditions) and the density field is allowed to evolve in the interior of the channel. We note that Dirichlet boundary conditions were also used in the simulations of *García-Villalba and del Álamo* [2011].

The flow is initialized first from rest and allowed to spin up until a fully developed turbulent velocity profile is obtained before the linear density stratification is imposed. Figure 2 shows the comparison of the unstratified velocity profile from the zero-equation RANS simulation to the unstratified DNS channel flow velocity profile of *García-Villalba and del Álamo* [2011] along with the initial density stratification. The velocity is normalized as $U^+=\bar{U}/u_\tau$ and the density is normalized as $\rho^+=(\bar{\rho}-\rho_{top})/(\rho_{bottom}-\rho_{top})$, where ρ_{top} and $\rho_{botttom}$ represent the density at the top and bottom boundaries of the channel, respectively. As it can be seen, the agreement between the RANS simulation and the DNS results are excellent.

4. Results and Discussion

In this section, the results of the RANS numerical simulations using different turbulent Prandtl number formulations are presented and discussed. We note that for the zero-equation model, the simulations use the modified turbulent viscosity (ν_t) proposed by *MA* as discussed in section 2. The results obtained from using the homogeneous Pr_t's given in equation (3) as well as the *MA* formulation for Pr_t given by equation (37) are compared with the modified Pr_t given by equation (20). The *MA* proposition is given by

$$Pr_t=Pr_{t0}\frac{(1+\beta Ri_g)^\alpha}{(1+\beta_\rho Ri_g)^{\alpha_\rho}}, \tag{37}$$

where $\beta=10$, $\alpha=-1/2$, $\beta_\rho=10/3$, and $\alpha_\rho=-3/2$ are empirical constants.

First, the fully developed velocity and density profiles using the zero-equation closure scheme and the homogeneous Pr_t formulations given by equations (3) and also (37) for $Ri_\tau=60$ are shown in Figures 3a and 3b. Superimposed on these plots are the profiles obtained from the DNS data of *García-Villalba and del*

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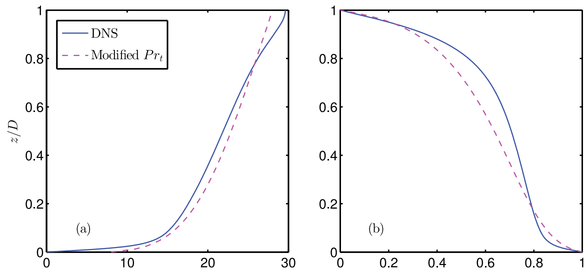

Figure 4. Comparisons of (a) velocity profiles and (b) density profiles, for $Ri_\tau = 120$ obtained from the zero-equation closure scheme using the modified proposition for Pr_t shown in equation (20) with the channel flow DNS data of *García-Villalba and del Álamo* [2011].

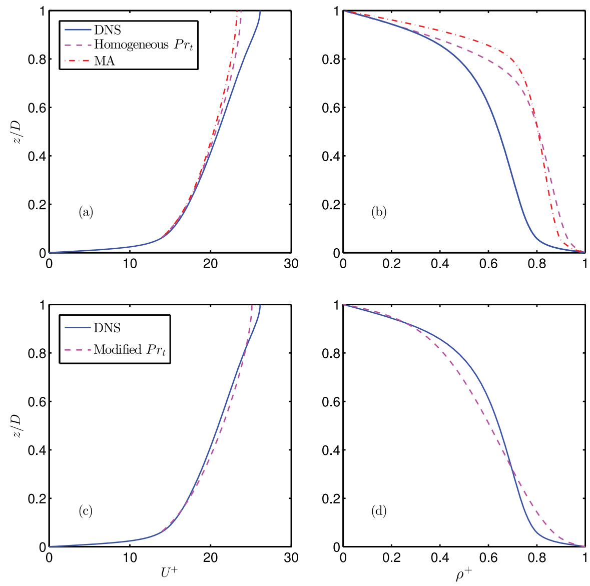

Figure 5. (left) Velocity profiles and (right) density profiles for $Ri_\tau = 60$ obtained from the standard k-ϵ closure scheme using (a, b) the Pr_t formulations given by equations (3) and (37) and (c, d) the modified Pr_t formulation (equation (20)) compared with the channel flow DNS data of *García-Villalba and del Álamo* [2011].

Álamo [2011]. It is evident that the homogeneous Pr_t formulations permit excessive mixing of the density and consequently a velocity profile similar to the unstratified case is obtained. On the other hand, the results of the RANS simulation using the modified Pr_t given by equation (20) show a much improved prediction of both momentum and scalar, especially a much closer agreement with the DNS density profile, thus highlighting the effect of the $(1-z/D)$ correction that has been proposed.

Furthermore, to better assess the robustness of the proposed correction, we make use of the modified Pr_t given by equation (20) to simulate a flow with a stronger stratification of $Ri_\tau = 120$. The results are shown in Figure 4 and compared with the channel flow DNS data. The formulation shows good prediction of the density and velocity profiles.

In order to assess the applicability of the modified Pr_t in more sophisticated RANS closure schemes, the channel flow simulation is performed using the standard k-ϵ model. Figure 5 shows the prediction of velocity and density profiles for $Ri_\tau = 60$ compared to the DNS profiles using the standard k-ϵ model for both the homogeneous Pr_t's given in equations (3) and (37) and the modified parameterization given by equation (20). Similar to the prediction shown in Figure 3, the homogeneous Pr_t formulations show overmixing of density and different velocity profiles compared to the DNS profile as shown in Figures 5a and 5b, respectively. Interestingly (but as expected), the modified proposition substantially improves the prediction of the density profile and consequently the velocity profile as can be seen in Figures 5c and 5d, respectively. This highlights the need for the proposed $(1-z/D)$ correction and the applicability of the modified Pr_t in more sophisticated turbulence models.

5. Concluding Remarks

In this study, we have investigated the use of homogeneous turbulent Prandtl number (Pr_t) parameterizations to predict the mixing of momentum and density in a stably stratified channel flow. We have made use of the stratified parabolic turbulent viscosity formulation proposed by *MA* as well as the standard k-ϵ model and tested the efficacy of the homogeneous Pr_t's in the context of one-dimensional RANS closure schemes. The comparison of the RANS results with data from DNS of stratified channel flow clearly shows the inadequacy of the homogeneous Pr_t formulations for correctly simulating a stratified channel flow. In order to account for the effect of the wall boundary, we have proposed a reasonable modification to the homogeneous Pr_t formulation by introducing a linear correction based on the distance from the wall $(1-z/D)$. This proposition was motivated in part by observed trends of the turbulent Prandtl number in unstratified wall-bounded flows from a number of studies such as those of *Kawamura et al*. [1998], *Crimaldi et al*. [2006], and *Sirinavasan and Papavassiliou* [2010] as well as some modeling reasoning. The RANS results using the modified formulations compare well with the DNS data, highlighting the utility of the proposed modification.

In essence, these results highlight the need to modify the homogeneous turbulent Prandtl number formulations for wall-bounded flows. To this end, the simple correction presented in this study has shown remarkable improvement in predicting both momentum and scalar mixing. We believe these findings will be useful in numerical modeling of many geophysical flows influenced by wall effects.

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