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On the turbulent Prandtl number in homogeneous stably stratified turbulence

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In this paper, we derive a general relationship for the turbulent Prandtl number Pr_t for homogeneous stably stratified turbulence from the turbulent kinetic energy and scalar variance equations. A formulation for the turbulent Prandtl number, Pr_t, is developed in terms of a mixing length scale L_M and an overturning length scale L_E, the ratio of the mechanical (turbulent kinetic energy) decay time scale T_L to scalar decay time scale T_ρ and the gradient Richardson number Ri. We show that our formulation for Pr_t is appropriate even for non-stationary (developing) stratified flows, since it does not include the reversible contributions in both the turbulent kinetic energy production and buoyancy fluxes that drive the time variations in the flow. Our analysis of direct numerical simulation (DNS) data of homogeneous sheared turbulence shows that the ratio $L_M/L_E \approx 1$ for weakly stratified flows. We show that in the limit of zero stratification, the turbulent Prandtl number is equal to the inverse of the ratio of the mechanical time scale to the scalar time scale, T_L/T_ρ. We use the stably stratified DNS data of Shih *et al.* (*J. Fluid Mech.*, vol. 412, 2000, pp. 1–20; *J. Fluid Mech.*, vol. 525, 2005, pp. 193–214) to propose a new parameterization for Pr_t in terms of the gradient Richardson number Ri. The formulation presented here provides a general framework for calculating Pr_t that will be useful for turbulence closure schemes in numerical models.

1. Introduction

The subject of turbulent mixing in stably stratified flows has received much attention. This is not surprising, since stably stratified flows are prevalent in the natural environment such as in the atmospheric boundary layer, the oceans, lakes and estuaries. The presence of buoyancy forces because of stratification has a substantial effect on the flow development and mixing processes and hence influences the distribution of substances such as pollutants, nutrients and suspended matter in the environment. The ability to predict mixing and dispersion in such flows has many practical applications. For example, the management of air quality in the atmospheric boundary layer requires accurate models for predicting how turbulence disperses pollutants released by industrial activities. In the ocean, the rates of vertical advection and diapycnal (across isopycnal) mixing are important to the dynamics of the thermocline and deep ocean. Excellent reviews of stably stratified geophysical

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flows have been given by Gregg (1987), Peltier & Caulfield (2003) and Ivey, Winters & Koseff (2008).

Turbulence closure schemes in Reynolds-averaged numerical models such as the k–ϵ model (Launder & Spalding 1972) make use of a turbulent Prandtl number $Pr_t = K_m/K_s$ to link the vertical momentum and scalar fluxes, where K_m and K_s are the momentum and scalar diffusivities respectively. For uni-directional shear flows, K_m and K_s are defined using the gradient-transport hypothesis as

$$K_m = -\frac{\overline{u'w'}}{\mathrm{d}\overline{u}/\mathrm{d}z}, \qquad K_s = -\frac{\overline{\rho' w'}}{\mathrm{d}\overline{\rho}/\mathrm{d}z}, \tag{1.1}$$

where $\overline{u'w'}$ is the Reynolds stress (turbulent momentum flux) and $\overline{\rho' w'}$ is the density flux. An overbar $\overline{(\,)}$ implies averaging (spatial or temporal). Data from direct numerical simulations (DNSs) and experiments have been used to develop parameterizations of Pr_t for stably stratified flows in terms of the gradient Richardson number Ri (e.g. Webster 1964; Launder 1975; Baum & Caponi 1992; Schumann & Gerz 1995) defined as

$$Ri = N^2/S^2, \tag{1.2}$$

where $N = \sqrt{(-g/\rho_0)(\mathrm{d}\overline{\rho}/\mathrm{d}z)}$ is the buoyancy frequency and $S = \mathrm{d}\overline{u}/\mathrm{d}z$ is the mean shear rate. The main drawback of calculating Pr_t using (1.1) from DNS data for stably stratified homogeneous flows stems from the fact that most of these flows are developing flows (i.e. locally non-equilibrium flows) and hence include reversible contributions to both the momentum and scalar fluxes in the production terms of the energy and scalar variance equations (see e.g. Komori *et al.* 1983; Rohr *et al.* 1988; Gerz, Schumann & Elghobashi 1989; Holt, Koseff & Ferziger 1992). An appropriate measure of Pr_t can be obtained by using only the irreversible terms in the turbulent kinetic energy and scalar variance balance equations to calculate the turbulent diffusivities. This will allow for a parameterization that is more generally applicable than previous formulations which do not necessarily separate out the reversible components in both the turbulent kinetic energy production and scalar fluxes. Furthermore, the value of Pr_t under neutral (unstratified) conditions is known to be close to unity, but there is no consensus on what the specific neutral value of Pr_t should be. Data from numerical simulations and experiments suggest values of Pr_t in the range 0.5–1.0 for neutrally stratified flows (Kays & Crawford 1993; Kays 1994).

In the current paper, we present a formulation for the turbulent Prandtl number for stably stratified flows, where the fluctuating quantities (such as velocity and scalar fluctuations) are statistically homogeneous. Statistical homogeneity requires the mean velocity gradients and mean density gradients to be uniform, although they can vary with time (e.g. Pope 2000). The layout of the paper is as follows. In § 2, we begin with a presentation of the evolution equations for the turbulent kinetic energy and density fluctuations to lay the context for our analysis followed by a derivation of the turbulent Prandtl number in terms of relevant length scales and time scales. In § 3, we provide an analysis of DNS results leading to a model for Pr_t that arises directly from our derivations. Conclusions are given in § 4.

2. Theoretical formulation for the turbulent Prandtl number

2.1. *Dynamical equations*

The evolution equations for the turbulent kinetic energy and scalar (density) fluctuations for a homogeneous stably stratified flow with the Boussinesq approximation can be written as

$$\frac{\partial(\overline{q^2}/2)}{\partial t} = -\overline{u'w'}\frac{\mathrm{d}\overline{u}}{\mathrm{d}z} - \frac{g}{\rho_0}\overline{\rho' w'} - \epsilon, \tag{2.1}$$

$$\frac{\partial(\overline{\rho'}^2/2)}{\partial t} = -\overline{\rho' w'}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} - \epsilon_\rho, \tag{2.2}$$

where $q^2 = (u'^2 + v'^2 + w'^2)$ is twice the turbulent kinetic energy per unit mass; $-\overline{u'w'}(\mathrm{d}\overline{u}/\mathrm{d}z)$ is the rate of production of turbulent kinetic energy; $(-g/\rho_0)(\overline{\rho' w'})$ is the buoyancy flux; and $\epsilon = \nu\overline{(\partial u_i/\partial x_j)(\partial u_i/\partial x_j)}$ and $\epsilon_\rho = \kappa\overline{(\nabla\rho')^2}$ are the dissipation rates of kinetic energy and scalar fluctuations, respectively. For a stationary state, (2.1) and (2.2) simplify to

$$-\overline{u'w'}\bigg/\frac{\mathrm{d}\overline{u}}{\mathrm{d}z} = \frac{g}{\rho_0}\overline{\rho' w'}\bigg/\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}z}\right)^2 + \epsilon\bigg/\left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}z}\right)^2, \tag{2.3}$$

$$-\overline{\rho' w'}\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}z} = \epsilon_\rho. \tag{2.4}$$

Equations (2.3) and (2.4) imply that the production terms are in balance with the dissipation terms. Therefore, in stationary flows, the productions terms represent the irreversible transfer of kinetic energy and density. In a non-stationary flow, the production terms are no longer in balance with the dissipation terms. However, the irreversible transfer of kinetic energy and density are still correctly represented by ϵ and ϵ_ρ.

2.2. *Turbulent Prandtl number*

For stationary flows, the vertical momentum eddy diffusivity can be obtained from (2.3) as

$$K_m = \epsilon/S^2 + Ri\,K_s, \tag{2.5}$$

where $K_s = -\overline{\rho' w'}/(\mathrm{d}\overline{\rho}/\mathrm{d}z)$ is the vertical scalar eddy diffusivity previously defined in (1.1). It also follows from (2.4) and (2.5) that

$$K_m = \frac{\epsilon + \epsilon_{PE}}{S^2}, \tag{2.6}$$

where $\epsilon_{PE} = N^2\epsilon_\rho(\mathrm{d}\overline{\rho}/\mathrm{d}z)^{-2}$ is the rate of dissipation of turbulent potential energy. We also note that (2.5) is equivalent to the flux Richardson number formulation for K_m used often in the literature (e.g. Gregg 1987), given by

$$K_m = \left(\frac{1}{1 - R_f}\right)\frac{\epsilon}{S^2}, \tag{2.7}$$

where the flux Richardson number R_f is defined as the ratio of the buoyancy flux $B = -g/\rho_0(\overline{\rho' w'})$ to the rate of production of turbulent kinetic energy $P = -\overline{u'w'}(\mathrm{d}\overline{u}/\mathrm{d}z)$. It should be noted that R_f calculated using this definition can attain negative values in non-stationary flows (see e.g. Shih *et al.* 2005).

We now consider how these results may be generalized for non-stationary flows described by (2.1) and (2.2). We start by noting that a diapycnal eddy diffusivity $K_s^* = \epsilon_\rho/(\mathrm{d}\overline{\rho}/\mathrm{d}z)^2$ is a measure of irreversible mixing of a scalar (density in this case) in both stationary and non-stationary homogeneous flows (see Winters & D'Asaro 1996; Venayagamoorthy & Stretch 2006). For stationary flows, K_s^* and K_s are equal (which follows from (2.4)). By analogy, we define an irreversible vertical momentum eddy diffusivity K_m^* as

$$K_m^* = \epsilon/S^2 + Ri\,K_s^*. \tag{2.8}$$

It follows from (2.1) that

$$\frac{1}{S^2}\frac{\partial(\overline{q^2}/2)}{\partial t} = (K_m - K_m^*) + Ri(K_s^* - K_s). \tag{2.9}$$

Note for stationary flows, $K_m^* = K_m$ and $K_s^* = K_s$ (which follow from (2.3) and (2.4) respectively). Conceptually, the production term $-\overline{u'w'}(\mathrm{d}\overline{u}/\mathrm{d}z)$ in the turbulent kinetic energy balance equation can be thought to consist of two parts: one part that is irreversible in the sense that at a given time it balances the total dissipation (kinetic and potential) as parameterized by K_m^*, while the other part is reversible in the sense that it accounts for the time variations in the energy balance equation. Similarly, the density flux term in the density variance equation can be thought to consist of an irreversible part that is in balance with ϵ_ρ as parameterized by K_s^* and a reversible part that accounts for the time rate of change of density variance. For example, reversible fluxes can be generated by linear internal waves in stratified flows. It follows from this reasoning that (2.6) with K_m replaced by K_m^* provides a measure of irreversible transfer of turbulent kinetic energy caused by momentum fluxes in a non-stationary stratified flow. Furthermore, this argument also implies that the flux Richardson number R_f should then be calculated as

$$R_f = \frac{\epsilon_{PE}}{\epsilon + \epsilon_{PE}}. \tag{2.10}$$

This form of the flux Richardson number was previously suggested by Peltier & Caulfield (2003); R_f calculated using (2.10) provides a measure of irreversible mixing and hence excludes reversible fluxes that are included if R_f is calculated as B/P, where B and P are the buoyancy flux and production of turbulent kinetic energy previously defined. The turbulent Prandtl number based on the irreversible contributions to the momentum and scalar fluxes may now be defined as

$$Pr_t^* = \frac{K_m^*}{K_s^*} = \frac{\epsilon}{S^2 K_s^*} + Ri. \tag{2.11}$$

We can test these ideas using the DNS data of homogeneous shear flows of Shih *et al.* (2000). These simulations are for temporally developing homogeneous turbulence with uniform shear and uniform stratification and for microscale Reynolds numbers $Re_\lambda \leqslant 90$. For small Ri the turbulent kinetic energy grows in time, while for large Ri the energy decays. A stationary state is attained for $Ri \simeq 0.17$. The stationary Ri depends on Re_λ and the initial non-dimensional shear rate Sq^2/ϵ which was equal to 4 for the cases considered here. Figure 1(a) shows Pr_t^* obtained using (2.11) as a function of non-dimensional time St for various values of Ri. It can be seen that Pr_t^* approaches a constant (but different) value for each Ri when $St > 6$ with no oscillations. Figure 1(b) shows a similar plot of Pr_t using the definitions

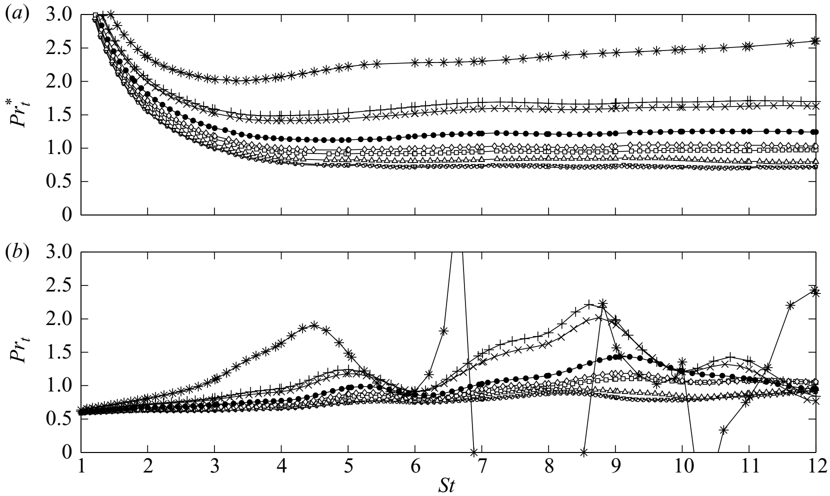

FIGURE 1. The turbulent Prandtl number as a function of non-dimensional time St for various gradient Richardson numbers Ri calculated from the DNS data of Shih *et al.* (2000). The data are shown for non-dimensional time $St \geqslant 1$: ○, $Ri = 0.05$; ▽, $Ri = 0.06$; △, $Ri = 0.1$; □, $Ri = 0.15$; ◇, $Ri = 0.18$; ●, $Ri = 0.25$; ×, $Ri = 0.37$; +, $Ri = 0.4$; ∗, $Ri = 0.6$. (*a*) $Pr_t^* = K_m^*/K_s^*$ (from (2.11)); (*b*) $Pr_t = K_m/K_s$ (from (1.1)). Note the oscillations in Pr_t for $Ri = 0.6$ have been truncated.

for K_m and K_s given by (1.1). In this case, as Ri increases, large oscillations appear because of reversible fluxes associated with internal waves. Note the short-time ($St < 2$) behaviour of Pr_t^* and Pr_t shown in figure 1 is an artefact of the initial conditions specified for these particular simulations, i.e. isotropic velocity field with no density fluctuations, and hence should be disregarded. Experimental results for stably stratified flows in the works of Rohr *et al.* (1988) and Webster (1964) also show large variations in Pr_t for a given Ri. Hence by distinguishing reversible from irreversible contributions in the energy and scalar balances, a formulation for the turbulent Prandtl number that is applicable to both stationary and non-stationary flows is obtained.

2.3. *Relevant length scales and time scales*

Here, we discuss the relevant length scales and time scales that will be used in § 3 to model the turbulent Prandtl number Pr_t^*. Using dimensional and physical arguments, a number of length scales have been used to characterize the dynamics of stratified turbulence. Two relevant length scales are a characteristic mixing length scale $L_M = (\overline{q^2})^{1/2}/S$ and an overturning length scale $L_E = (\overline{\rho'^2})^{1/2}/|(\mathrm{d}\overline{\rho}/\mathrm{d}z)|$; L_M represents a rough measure of the size of active turbulent fluctuations in momentum and can be identified as an approximate measure of the average eddy size; L_E is the well-known Ellison length scale, and it provides a measure of the vertical distance travelled by particles before either returning towards their equilibrium levels or mixing in a density field (Ellison 1957). It is expected that the ratio of these length scales $L_M/L_E \approx 1$ for energetic flows. Figure 2 shows a plot of L_M/L_E as a function of non-dimensional time St for $0 \leqslant Ri \leqslant 0.6$, from the DNS data of Shih *et al.* (2000). The data suggest that L_M/L_E is approximately equal to unity (i.e. $L_M/L_E \simeq 1.05$) for weakly stratified

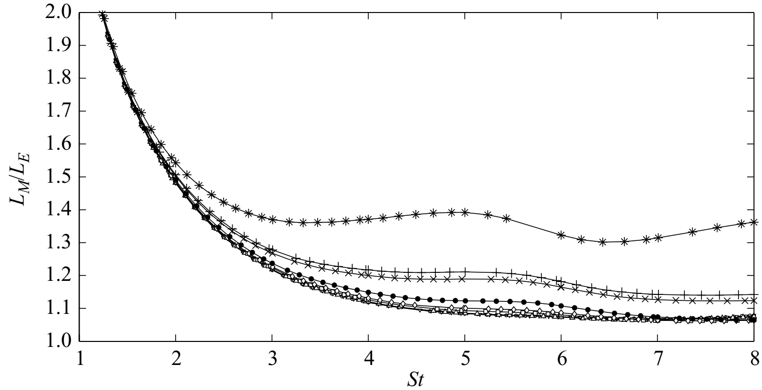

FIGURE 2. The mixing length scale L_M normalized by the Ellison length scale L_E as a function of non-dimensional time St for various gradient Richardson numbers Ri calculated from the DNS data of Shih *et al.* (2000). The data are shown for non-dimensional time $St \geqslant 1$: ○, $Ri = 0.05$; ▽, $Ri = 0.06$; △, $Ri = 0.1$; □, $Ri = 0.15$; ◇, $Ri = 0.18$; ●, $Ri = 0.25$; ×, $Ri = 0.37$; +, $Ri = 0.4$; ∗. $Ri = 0.6$.

flows (i.e. for $Ri \lesssim 0.25$). The data also show that for strongly stable flows ($Ri \gtrsim 0.25$), L_M/L_E is an increasing function of Ri. Using the definitions of L_M and L_E, it can be shown that L_M/L_E is given by

$$\frac{L_M}{L_E} = \left(\frac{E_{KE}}{E_{PE}}\right)^{1/2} Ri^{1/2}, \tag{2.12}$$

where $E_{KE} = (1/2)\overline{q^2}$ is the turbulent kinetic energy per unit mass and $E_{PE} = -(1/2)(g/\rho_0)(\mathrm{d}\overline{\rho}/\mathrm{d}z)^{-1}\overline{\rho'^2}$ is the (available) turbulent potential energy. For now, if we assume that the ratio of the turbulent kinetic energy to the potential energy is constant for large Ri, then $L_M/L_E \propto Ri^{1/2}$. We shall later show in § 3.1 that this is indeed a valid assumption for large Ri.

An important parameter widely used in second-moment closure models (see e.g. Pope 2000) is the mechanical to scalar time scale ratio $T_L/T_\rho = 2\gamma$; $T_L = E_{KE}/\epsilon$ is the turbulent kinetic energy decay time scale and $T_\rho = ((1/2)\overline{\rho'}^2)/\epsilon_\rho$ is the scalar decay time scale. Venayagamoorthy & Stretch (2006) and authors' unpublished observations showed that T_L/T_ρ is relatively insensitive to Ri with $\gamma \simeq 0.7$, using DNS results and experiment data. Their dataset included DNS results of homogeneous sheared stably stratified turbulence of Shih *et al.* (2000), unstratified homogeneous sheared DNS data of Rogers, Mansour & Reynolds (1989) and experimental data on grid-generated turbulence (Srivat & Warhaft 1983; Itsweire, Helland & Atta 1986; Yoon & Warhaft 1990; Mydlarski 2003). The insensitivity of the mechanical-to-scalar time scale ratio to stratification has important simplifying implications for modelling the turbulent Prandtl number.

3. Modeling the turbulent Prandtl number

In this section, we propose a model for Pr_t^* as a function of the gradient Richardson number Ri and use DNS data to support it.

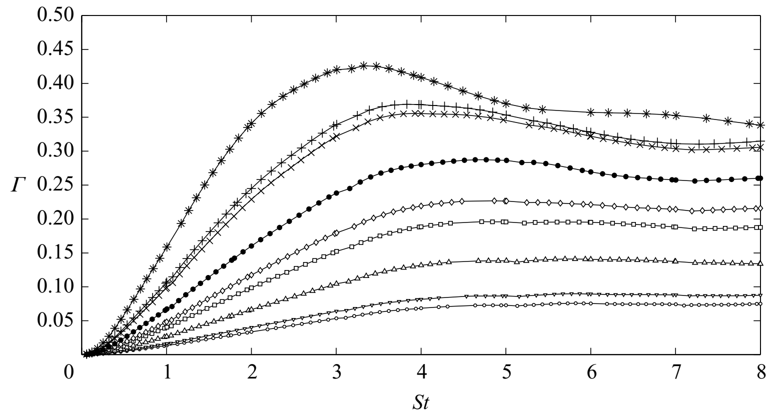

FIGURE 3. The mixing efficiency Γ as a function of non-dimensional time St for various gradient Richardson numbers, Ri, calculated from the DNS data of Shih *et al.* (2000). The data are shown for non-dimensional time $St \geqslant 0$: $\circ$, $Ri = 0.05$; ∇, $Ri = 0.06$; $\triangle$, $Ri = 0.1$; $\square$, $Ri = 0.15$; $\diamond$, $Ri = 0.18$; $\bullet$, $Ri = 0.25$; $\times$, $Ri = 0.37$; $+$, $Ri = 0.4$; $*$, $Ri = 0.6$.

3.1. *Model formulation*

Equation (2.11) can be recast in terms of the length scales and time scales discussed in § 2.3 as

$$Pr_t^* = \frac{1}{2\gamma}\frac{L_M^2}{L_E^2} + Ri, \tag{3.1}$$

where $\gamma = (1/2)(T_L/T_\rho)$. At $Ri = 0$, $Pr_t^* = ((1/2)\gamma)/(L_M^2/L_E^2)$. The DNS data of Rogers *et al.* (1989) for neutrally stratified homogeneous shear flow show that $L_M/L_E \simeq 1$. Moreover, it can be observed from figure 2 that $L_M/L_E \simeq 1$ for $Ri \lesssim 0.25$. Hence the neutral value of the turbulent Prandtl number is $Pr_{t0} = 1/(2\gamma)$ to a very good approximation. In other words, the neutral turbulent Prandtl is approximately equal to the ratio of the scalar time scale to mechanical time scale, T_ρ/T_L. Using $\gamma \simeq 0.7$ as suggested by Venayagamoorthy & Stretch (2006) gives a neutral value for Pr_{t0} equal to 0.7. It is worth noting that Townsend (1976) used rapid distortion theory (RDT) to show that the neutral value of the turbulent Prandtl number Pr_{t0} reaches a value of 0.7 at a non-dimensional time $St \simeq 5$.

The near constancy of γ and L_M/L_E for low Ri provides a model for Pr_t^* given by

$$Pr_t^* = \frac{1}{2\gamma} + Ri = Pr_{t0} + Ri, \tag{3.2}$$

which is valid in the weakly stratified regime ($Ri \lesssim 0.25$). Equation (3.2) indicates that Pr_t^* is a linear function of Ri with a slope of 1. On the other hand, at high Ri, the DNS data indicate that L_M/L_E is a strong function of Ri (see figure 2). Equating (3.1) with $Pr_t^* = Ri/R_f$ (which follows from (2.10) and (2.11)) allows us to write L_M^2/L_E^2 as a function of Ri, namely

$$\frac{L_M^2}{L_E^2} = \frac{2\gamma}{\Gamma}Ri, \tag{3.3}$$

where $\Gamma = R_f/(1 - R_f) = \epsilon_{PE}/\epsilon$ is the instantaneous mixing efficiency of the flow with R_f defined by (2.10). Γ provides a measure of the irreversible conversion of kinetic energy into potential energy. Figure 3 shows a plot of Γ as function of St for various

Ri values. It is evident that Γ becomes approximately constant for large Ri. It is interesting to note that the overall mixing efficiency obtained from time-integrated calculations for decaying homogeneous stably stratified turbulence also indicate an approximately constant value at large Ri (see Stretch *et al.* 2010). By combining (2.12) and (3.3), it follows that the ratio of turbulent kinetic energy to potential energy is given by

$$\frac{E_{KE}}{E_{PE}} = \frac{2\gamma}{\Gamma}. \tag{3.4}$$

The ratio E_{KE}/E_{PE} is indeed approximately a constant for large Ri, since γ has been shown to be relatively insensitive to stratification effects (Venayagamoorthy & Stretch 2006). Therefore the near constancy of Γ or equivalently R_f provides a model for Pr_t^* in the large-Ri limit given by

$$Pr_t^* = \frac{1}{R_{f\infty}} Ri, \tag{3.5}$$

where $R_{f\infty}$ is the asymptotic value of the flux Richardson number for large Ri. This means that Pr_t^* is again a linear function of Ri but with a slope equal to $1/R_{f\infty}$. Equations (3.2) and (3.5) provide the low-Ri and high-Ri behaviour of Pr_t^*. All that remains to be done is to fit a blending function for Pr_t^* as a function of Ri that smoothly transitions between these two limits. Before we proceed to the curve fitting, it is worth mentioning in passing that Launder (1975) provided a model for Pr_t obtained using second-moment closure models that capture the above-mentioned behaviour of Pr_t for the low-Ri limit predicted by our formulation (i.e. a linear dependence on Ri with a slope of 1). However, his model suggests that in the strongly stratified limit, Pr_t approaches a constant value of 2. From (3.5), this implies that the mixing efficiency increases linearly with Ri in this limit, which contradicts the DNS data presented in figure 3.

3.2. *Model for the turbulent Prandtl number*

We find that the empirical formulation given by Schumann & Gerz (1995) is a suitable blending function that captures the correct behaviour of Pr_t^* in the large-Ri limit. This is not surprising, since they made the assumption that R_f approaches a constant limit for large Ri, but they did not have access to sufficient data to show that this was indeed a reasonable assumption. However, their assumption that Pr_t is independent of Ri as $Ri \to 0$ is not consistent with (3.2). We propose the following revised function that captures the behaviour of Pr_t^* for both Ri limits to first-order accuracy:

$$\frac{Pr_t^*}{Pr_{t0}} = \exp\left(-\frac{Ri}{Pr_{t0}\Gamma_\infty}\right) + \frac{Ri}{R_{f\infty}Pr_{t0}}, \tag{3.6}$$

where it is important to note that $R_{f\infty}$ is obtained using (2.10) and the neutral Prandtl number $Pr_{t0} \simeq 1/(2\gamma)$. Note also that $\Gamma_\infty = R_{f\infty}/(1 - R_{f\infty})$.

The DNS results shown in figure 3 indicate an average value of $\Gamma_\infty \simeq 1/3$, which translates to a flux Richardson of $R_{f\infty} = 1/4$. This seems to be a reasonable upper bound for R_f that is in agreement with the time-integrated mixing efficiency at large Ri of the decaying simulations of Stretch *et al.* (2010). Figure 4 shows the prediction for Pr_t^* given by (3.6) as a function of Ri. Also shown are values of Pr_t^* computed from the DNS results by averaging over the time interval $St = 6$ to $St = 8$ (see figure 1). It is seen that with a mixing efficiency $\Gamma_\infty = 1/3$ and a mechanical-to-scalar time

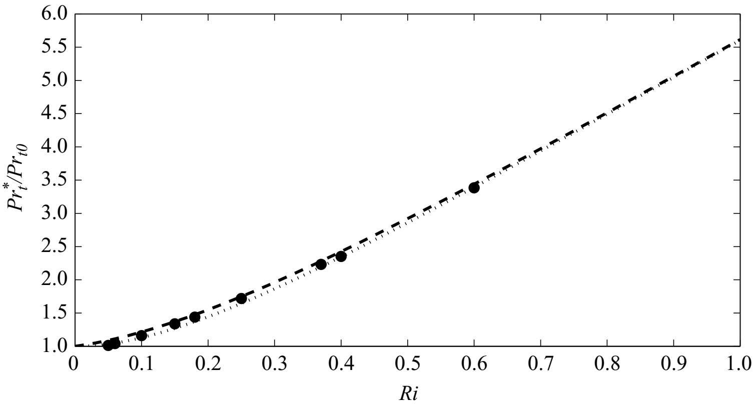

FIGURE 4. The turbulent Prandtl number Pr_t^* as a function of gradient Richardson number Ri: ●, DNS data (Shih *et al.* 2000); dashed line, model prediction given by (3.6); dotted line, curve fit of Schumann & Gerz (1995).

scale ratio, T_L/T_ρ, of 1.4, the model agrees remarkably well with the data. The empirical function of Schumann & Gerz (1995) is also shown, and it is evident that the differences are minor for all practical purposes, although the agreement seems fortuitous, since their formulation (and many other formulations) for Pr_t is obtained by fitting a curve through very noisy data.

4. Concluding remarks

In this study, our main proposition is the separation of reversible and irreversible contributions that inherently arise in the kinetic energy production and buoyancy flux terms in the balance equations for non-stationary homogeneous stably stratified shear flows. We have argued that the turbulent Prandtl number should be calculated using the dissipation terms ϵ and ϵ_ρ, since these terms represent the irreversible transfers of momentum and scalar fluxes.

Using this novel proposition, we have derived a formulation for the turbulent Prandtl number in terms of relevant length scales, time scales and the gradient Richardson number. The turbulent Prandtl number formulation we have presented is more general than previous formulations in that it is not restricted to stationary homogeneous flows. We have exploited the detailed information available from DNS of stably stratified homogeneous turbulence to infer the behaviour of the turbulent Prandtl number for weakly and strongly stratified flows and have then formulated a model for the turbulent Prandtl number as a function of Ri. We note that new DNS with an expanded Reynolds number range is necessary to check the validity of the relationships between the length scales L_M and L_E, time scales T_L and T_ρ and the mixing efficiency for large Ri. A natural extension of this work is to also explore the applicability of the formulation presented for Pr_t^* to inhomogeneous stratified flows. The results reported here may be applicable to the development of improved turbulence models including subgrid scale parameterizations for large-eddy simulations of stably stratified turbulent flows.

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