CN 11 Unsteady Flow in Open Channels

This Chapter probes deeper into the one-dimensional (1-D) analysis of floodwave propagation. We derive the governing equation for floodwave propagation in Section 11.1, and seek solutions to the advection-dispersion equation in Section 11.2. The topic of unsteady flow is also covered in Liggett and Cunge (1975), Abbott and Basco (1989), Fennema and Chaudhry (1990), Singh (1997), Sturm (2001), Ponce (2014), Battjes and Labeur (2017), and Palu and Julien (2020).

11.1. Floodwave Propagation Equation

Three relationships describe unsteady flow in open channels: (1) conservation of mass in Section 11.1.1; (2) flow resistance in Section 11.1.2; and (3) momentum in Section 11.1.3. They combine into a diffusion equation in Section 11.1.4.

11.1.1. Continuity for Unsteady Flow

The principle of conservation of mass indicates that the mass of water remains constant. In Figure 11.1, we identify the top width W, wetted perimeter P, and the flow discharge Q is the product of mean flow velocity V and cross section area A. We can add complexity with rainfall intensity i_r, infiltration i_b through the wetted perimeter and lateral inflow q_l, (flow discharge per unit width).

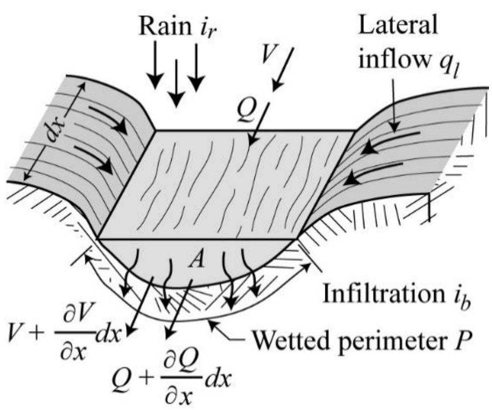

Figure 11.1. Continuity for open channels

The total volume of water in the control volume is Adx. Over a reach length dx, the discharge Q enters the control volume and the discharge leaving the control volume is $Q+\frac{dQ}{dx}dx$. When including rain and lateral inflow while losing water through infiltration, the total volumetric fluxes equal the internal volumetric change

$$Q+q_l dx+i_r W dx-i_b P dx-\left(Q+\frac{\partial Q}{\partial x}dx\right)=\frac{\partial (Adx)}{\partial t}\ .$$

We divide by dx and reduce to

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=i_r W+q_l-i_b P\ .$$

Of course, when rainfall precipitation, infiltration and lateral flow are negligible, we obtain the main relationship describing continuity, or conservation of mass in rivers

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \qquad (11.1)$$

For rectangular channels of constant width W, it reduces further to

$$\frac{\partial h}{\partial t}+\frac{1}{W}\frac{\partial Q}{\partial x}=0 \quad \text{or} \quad \frac{\partial^2 h}{\partial t\partial x}=-\frac{1}{W}\frac{\partial^2 Q}{\partial x^2} \qquad (11.1a)$$

11.1.2. Flow Resistance

Resistance to flow in open channels is described by Manning's equation

$$Q=A\frac{m}{n}R_h^{2/3}S_f^{1/2}=k\,S_f^{1/2} \text{ where } k=A\frac{m}{n}R_h^{2/3}$$

In SI, $m=1$ and k is the conveyance coefficient. For wide-rectangular channels, we can simply write the discharge per unit width as a power function of flow depth

$$q=\frac{Q}{W}=Vh=\frac{m}{n}h^{5/3}S_f^{1/2}=\alpha h^{\beta} \text{ where } \alpha=\frac{m}{n}S_f^{1/2} \text{ and } \beta=5/3$$

The advantage of this formulation for wide-rectangular channels is that α and β remain constant while k varies with flow depth, hence

$$k=\frac{Q}{\sqrt{S_f}}=\frac{W}{\sqrt{S_f}}\alpha h^{\beta} \text{ or } S_f=\frac{Q^2}{k^2} \qquad (11.2)$$

and with constant values of W, α and S_f, we obtain

$$\frac{\partial k}{\partial h}=\frac{W}{\sqrt{S_f}}\alpha\beta h^{\beta-1}=\frac{\beta k}{h}$$

And because k is only a function of h, we can combine with Eq. (11.1a) to get

$$\frac{\partial k}{\partial t}=\frac{\partial k}{\partial h}\frac{\partial h}{\partial t}=\frac{\beta k}{h}\left(-\frac{1}{W}\frac{\partial Q}{\partial x}\right) \qquad (11.2a)$$

We can examine the time derivative of $S_f=Q^2/k^2$ when both Q and k vary with time. We have the derivative of a ratio like $\frac{d}{dt}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}$, which gives

$$\frac{\partial S_f}{\partial t}=\frac{\partial}{\partial t}\left(\frac{Q^2}{k^2}\right)=\frac{2Q}{k^2}\frac{\partial Q}{\partial t}-\frac{2Q^2}{k^3}\frac{\partial k}{\partial t} \qquad (11.2b)$$

which is combined with Eq. (11.2a) to give

$$\frac{\partial S_f}{\partial t}=\frac{2Q}{k^2}\frac{\partial Q}{\partial t}-\frac{2Q^2}{k^3}\left[\frac{\beta k}{h}\left(-\frac{1}{W}\frac{\partial Q}{\partial x}\right)\right] \qquad (11.2c)$$

The conveyance relationship only includes advection terms in $\frac{\partial Q}{\partial t}$ and $\frac{\partial Q}{\partial x}$, which corresponds to pure wave translation without deformation.

11.1.3. Momentum

We learned from Eq. (10.1) that

$$S_f = S_0 - (1 - Fr^2)\frac{\partial h}{\partial x} = S_0 - \Omega\frac{\partial h}{\partial x} \qquad (11.3)$$

where $\Omega = 1 - Fr^2$, and the Froude number Fr remains essentially constant at different flow depths. Taking the time derivative gives

$$\frac{\partial S_f}{\partial t} = -\Omega\frac{\partial^2 h}{\partial x \partial t} \qquad (11.3a)$$

We are now ready to derive the unsteady flow equation for open channels.

11.1.4. Flood Routing in Open Channels

Equations (11.1 to 11.3) describe unsteady flow in wide-rectangular channels.

Continuity $\quad \frac{\partial h}{\partial t} = -\frac{1}{W}\frac{\partial Q}{\partial x} \qquad$ (11.1a)

Conveyance $\quad S_f = \frac{Q^2}{k^2} \qquad$ (11.2)

Momentum $\quad S_f = S_0 - \Omega\frac{\partial h}{\partial x} \qquad$ (11.3)

The last equation (11.3) is also called the diffusive wave approximation, for a reason we are about to discover. The strategy adopted to solve these differential equations is to eliminate h from Eqs. (11.1a) and (11.3). This is done through differentiating Eq. (11.1) in space x and differentiating Eq. (11.3) in time t. Thus, combining Eqs. (11.1a) and (11.3a) and comparing with Eq. (11.2c) gives

$$\frac{\partial S_f}{\partial t} = \frac{2Q}{k^2}\frac{\partial Q}{\partial t} + \frac{2Q^2}{k^3}\left(\frac{\beta k}{hW}\frac{\partial Q}{\partial x}\right) = \frac{\Omega}{W}\frac{\partial^2 Q}{\partial x^2}$$

The attentive reader will notice here that the diffusion term $\partial^2 Q / \partial x^2$ stems from the momentum equation via Eq. (11.3a). Algebraic simplifications yield

$$\frac{\partial Q}{\partial t} + \beta V\frac{\partial Q}{\partial x} = \frac{\Omega Q}{2WS_f}\frac{\partial^2 Q}{\partial x^2} \qquad (11.4)$$

This basic relationship describes unsteady flow propagation in a wide-rectangular channel. This advection-diffusion (or advection-dispersion) equation is

$$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = K\frac{\partial^2 Q}{\partial x^2} \qquad (11.5)$$

where $c = \beta V$ is the flood celerity, and $K = \frac{\Omega Q}{2WS_f}$ is the flood diffusion coefficient.

We learn that the celerity of the flood wave in open channels is faster than the flow velocity because $c = \beta V$ and $\beta = 5/3$ in wide-rectangular channels from Manning's equation. The second important characteristic of the flood wave propagation equation is that the diffusion coefficient describes the attenuation of the flood wave as shown in Figure 11.2.

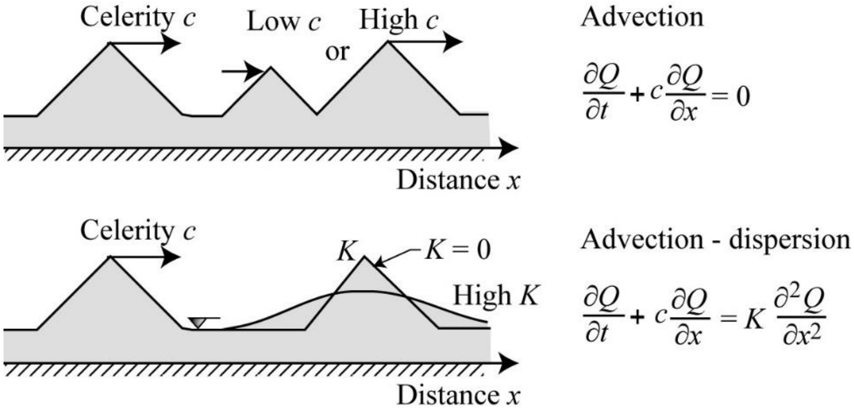

Figure 11.2. Floodwave propagation in wide open channels

The value of K increases when the discharge Q increases and the slope decreases. The flood waves of large flat rivers attenuate greatly in comparison with smaller floods in steep mountain channels, as discussed in Julien (2018). The term $\Omega = 1 - Fr^2$ in parameter K also indicates that flood wave attenuation increases when rivers have a low Froude number.

11.2. Floodwave Propagation Calculation

We explore an analytical solution for floodwave propagation in Section 11.2.1 followed with a numerical solution in Section 11.2.2. Useful references include Woolhiser (1975), Liggett and Cunge (1975), Chapra (1997), Woo et al. (2015), Chanson (2004) and Chaudhry (2008).

11.2.1. Analytical Solution for Flood Wave Propagation

The propagation of floodwaves in open channels can be analyzed by solving the advection-dispersion Eq. (11.5), where c in m/s is the flood wave celerity and K in m^2/s is the dispersion coefficient. For a constant pulse of water at a discharge Q_o over a duration T, the discharge $Q(x,t)$ is calculated at a distance x downstream from the source as a function of time t in a river given the mean flow celerity c as

$$Q(x,t)=\frac{Q_0}{2}\left\{erfc\left[\frac{x-ct}{2\sqrt{Kt}}\right]-erfc\left[\frac{x-c(t-T)}{2\sqrt{K(t-T)}}\right]\right\}+ \frac{Q_0}{2}e^{\frac{cx}{K}}\left\{erfc\left[\frac{x+ct}{2\sqrt{Kt}}\right]-erfc\left[\frac{x+c(t-T)}{2\sqrt{K(t-T)}}\right]\right\} \quad (11.6)$$

where $erfc(x)=1-erf(x)$ is the complementary error function from the error function $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\alpha^2}d\alpha$. Figure 11.3 plots the normal distribution, the error function and the complementary error function. Error functions are calculated with any mathematical package (e.g. the functions erf.precise and erfc.precise in Xcel). Example 11.1 analytically calculates the propagation of a flow pulse.

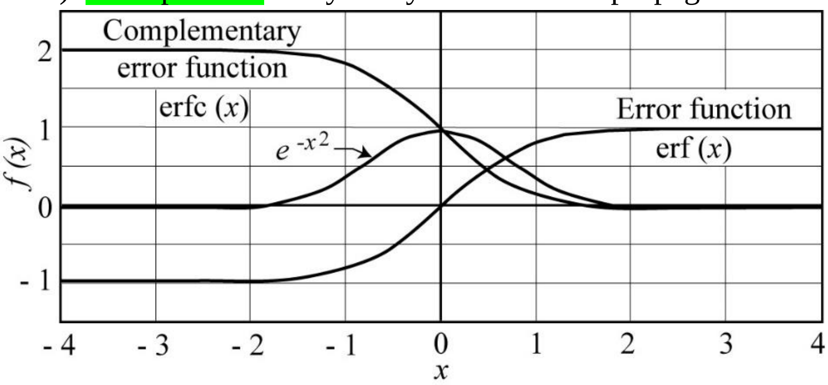

Figure 11.3. Normal, error and complementary error functions

Example 1.1: Analytical Solution for a Flow Pulse

A calculation example for the propagation of a flow pulse lasting T = 6 hours at an initial discharge Q_0 = 1,000 m^3/s given the mean flow celerity c = 1 m/s and dispersion coefficient K = 1,000 m^2/s is shown in Figure E-11.1. Let us calculate the discharge at a distance of 75 km after one day.

Solution

Consider $x = 75{,}000m$, $t = 86{,}400s$ and pulse duration $T = 6 \times 3{,}600 = 21{,}600s$.

From Eq. 11.6, we obtain

$$Q(75km,1day) = \frac{1{,}000}{2}\left\{erfc\left[\frac{-11{,}400}{18{,}590}\right] - erfc\left[\frac{10{,}200}{16{,}099}\right]\right\} +$$

$$\frac{1{,}000}{2} \times 3.733 \times 10^{32}\left\{erfc\left[\frac{161{,}400}{18{,}590}\right] - erfc\left[\frac{139{,}800}{16{,}099}\right]\right\}$$

$$Q(75km,1day) = [500(1.6142 - 0.3703)] + [500 \times 3.733 \times 10^{32}(1.1873 \times 10^{-34} - 1.1574 \times 10^{-34})]$$

$$Q(75km,1day) = 622 + 0.56 = 622.5cms$$

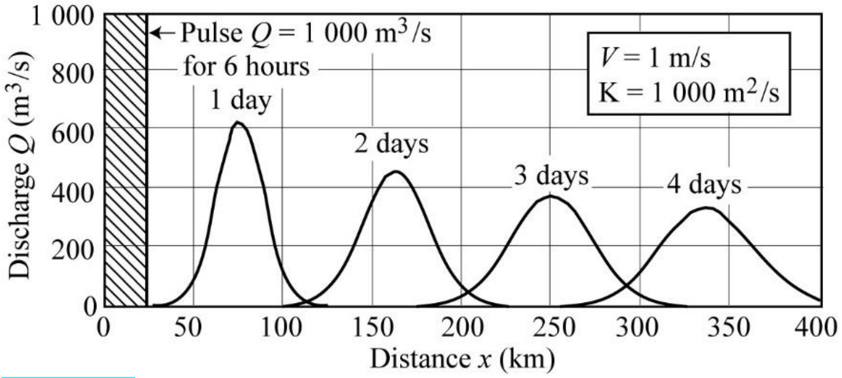

Fig. E-11.1. Analytical advection-dispersion example

The main characteristics of flood wave propagation are clearly visible from Fig. E-11.1: (1) translation of the floodwave moving downstream at the celerity $c = 1m/s = 86.4km/day$; and (2) floodwave attenuation through the parameter $K = \frac{\Omega Q}{2WS_f} \simeq \frac{(1 - Fr^2)Q}{2WS_f}$ as the flood propagates downstream. It is noted that the dispersion of the flood wave is due to the momentum equation Eq. (11.3) because $K = 0$ when $\Omega = 0$. Also, the principle of superposition can be applied to a sequence of step functions because Eq. (11.5) is linear. The advantage of the analytical solution is that we can directly calculate the values of discharge at any time and space value. However, the analytical solution becomes less practical for long hydrographs where discharge varies rapidly with time. To handle large variability in discharge, the numerical method of Section 11.2.2 is usually more convenient.

11.2.2. Numerical Solution for Flood Wave Propagation

The numerical solution to the advection-diffusion equation can contaminate the results by adding numerical diffusion which artificially attenuates the flood wave. Higher order numerical schemes can eliminate numerical diffusion (Abbott and Basco 1989). From Julien (2018), the grid size Δx and time step Δt are determined from the flood celerity c and diffusion coefficient K as $\Delta x = \frac{10K}{c}$, and $\Delta t = \frac{10K}{c^2}$. A practical finite difference numerical scheme of Eq. (11.5) without numerical diffusion is

$$Q_j^{k+1} = 0.1Q_{j-2}^k + 0.8Q_{j-1}^k + 0.1Q_j^k \quad (11.7)$$

The subscript from *j-2* to *j* refers to space, the superscript refers to time from *k* to *k+1*. This algorithm requires two upstream boundary conditions at *j-2* and *j-1* as shown in Figure 11.4, and the initial condition at *k* =0 describes the flow discharge along the channel reach at the beginning of the flood. Example 11.2 shows calculations for a double pulse, and Example 11.3 presents the case study of a dam break event in the Doce River.

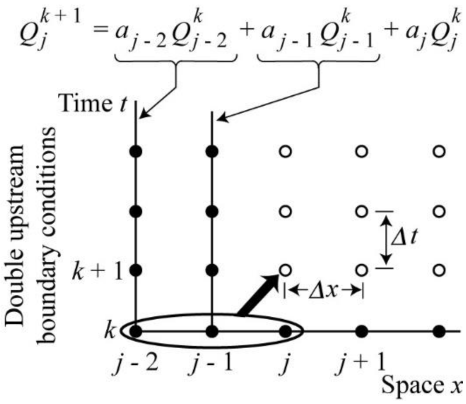

Figure 11.4. Double upstream boundary condition

Example 11.2: Numerical Solution for Triangular Pulses

Simulate the propagation of a double triangular flow pulse in a river where the celerity is c = 1 m/s and the diffusion coefficient is K = 1,000 m^2/s. Find the hydrograph at x = 100 km downstream of the double pulse shown in Figure E-11.2.

Solution:

First the grid spacing is $\Delta x = 10K/c = 10{,}000m = 10\,km$ and the time step $\Delta t = 10K/c^2 = 10{,}000\ s = 0.116\,day$. The algorithm is $Q_j^{k+1} = 0.1Q_{j-2}^k + 0.8Q_{j-1}^k + 0.1Q_j^k$ and we develop a marching procedure shown in Figure E-11.2. The boundary conditions are in the first two columns. Note that we offset the upstream boundary condition with a one time-step lag (because $c\Delta t = \Delta x$) in the downstream direction (see the table at x = -10 km and x = 0). For example, the discharge at successive times where x = 10 km are calculated as

At t = 0.116 day, $Q_{10km}^{0.116day} = (0.1\times 500) + (0.8\times 250) + (0.1\times 100) = 260cms$,

At, t = 0.231 day, $Q_{10km}^{0.231day} = (0.1\times 750) + (0.8\times 500) + (0.1\times 260) = 501cms$, etc.

	c m/s	1										
	K m2/s	1000			initial and upstream boundary conditions							
					discharge at x = 100 km							
	dx	10000	m									
	dt	10000	s									
time		distance x in km										
day	-10	0	10	20	30	40	50	60	70	80	90	100
0	500	250	100	100	100	100	100	100	100	100	100	100
0.116	750	500	260	115	100	100	100	100	100	100	100	100
0.231	1000	750	501	270	128	102	100	100	100	100	100	100
0.347	1250	1000	750.1	503	279	140	104	100	100	100	100	100
0.463	1000	1250	1000	750	505	287	150	107	101	100	100	100
0.579	750	1000	1200	1000	751	508	295	159	111	101	100	100
0.694	500	750	995	1160	995	751	511	303	168	115	102	100
0.81	250	500	749.5	987	1127	987	752	514	310	176	119	103
0.926	100	250	500	748	977	1099	978	752	517	317	184	123
1.042	300	100	260	500	746	967	1075	967	751	521	324	191
1.157	500	300	136	268	500	744	955	1053	956	749	524	330
1.273	700	500	303.6	166	278	502	741	944	1034	945	748	527
1.389	500	700	500.4	309	191	289	503	737	933	1016	934	745
1.505	300	500	660	501	317	212	301	505	733	921	999	924
1.62	100	300	496	628	499	325	232	312	507	729	910	984
1.736	100	100	299.6	490	602	494	333	249	324	510	725	900
1.852	100	100	120	299	482	580	489	341	265	335	513	721
1.968											346	516
2.083											293	356
2.199											361	305
2.315											466	366
2.431											505	461
2.546											433	494
2.662											306	426
2.778											195	306
2.894											135	199
3.009											111	138
3.125											103	113
3.241											101	104
3.356											100	101
3.472	100	100	100	100	100	100	100	100	100	100	100	100

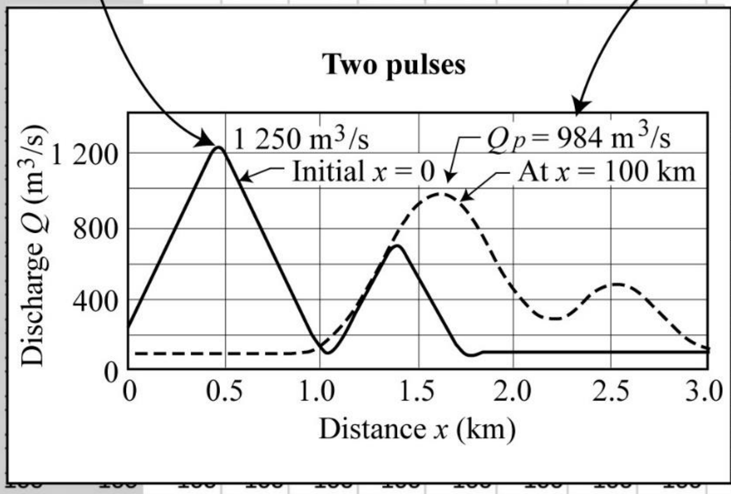

Fig. E-11.2. Numerical calculation table for floodwave propagation