Exact Procedure for Einstein-Johnson's Sidewall Correction in Open Channel Flow

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ABSTRACT: The classic sidewall correction procedure, Vanoni-Brook's method, originates from Einstein-Johnson's work. However, Johnson's contribution is almost oblivious in recent literature because his friction equation for the sidewall flow is difficult to solve. This note then revisits Einstein-Johnson's sidewall correction procedure and solves Johnson's friction equation explicitly and exactly. Besides, it extends Einstein-Johnson's procedure to transitional-, and rough-sidewall turbulent flows. The presented exact procedure facilitates Einstein-Johnson's sidewall correction for applications in flume experiments and sediment transport analysis in rivers, streams, and canals.

CE Database subject headings: Flume experiments; Friction factor; Lambert function; Open channel flow; Shear stress; Sidewall correction.

Introduction

Open channel flow has different shear stresses between the bed and the sidewalls. For rivers, the bed shear stress governs bedload transport thereby determining the channel vertical processes; the sidewall shear stress erodes the banks thereby influencing the bank stability. For flume experiments on sediment transport (Wang and Parker 2005; Heyman *et al*. 2015), bridge scour (Day and Raikar 2005), and vegetation flow (Cheng and Nguyen 2011), the sidewall correction is required for almost all tests. Otherwise, data from different flumes do not generally match, and data from narrow flumes (3D flow) cannot be used for wide rivers (2D flow). Therefore, the sidewall correction is interesting for both engineers and researchers.

Although several sidewall correction methods (Knight *et al*. 1984; Yang and Lim 1997; Guo and Julien 2005; Guo 2015) are available, Einstein's (1934, 1942) procedure or its modifications (Johnson 1942; Vanoni and Brooks 1957) are still widely used. Einstein hypothesized that: (i) the entire cross-section is divided into two sub-areas A_b and A_w which correspond to the bed and the sidewalls, respectively, where the fluid interfaces are frictionless so that the energy loss of each sub-area is dissipated by the associated wall shear stress; (ii) the two sub-areas are like two independent parallel channels with the same velocity V and energy slope S_f as the entire cross-section but different roughness; similar to the whole flow, Manning's resistance

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equation applies for both sub-areas. For the sidewall flow, these hypotheses lead to (Guo 2015):

$$R_w = \left(\frac{n_w V}{\sqrt{S_f}}\right)^{3/2}, \ A_w = 2hR_w, \ \ \tau_w = \gamma R_w S_f \tag{1}$$

where R_w, n_w, and τ_w are the sidewall hydraulic radius, Manning's coefficient, and shear stress, respectively; h is the flow depth; and γ is the specific water weight. Consequently, the bed sub-area flow has:

$$A_b = bh - A_w, \ \ R_b = A_b/b, \ \ \tau_b = \gamma R_b S_f \tag{2}$$

where b is the channel width; R_b and τ_b are the bed hydraulic radius and the shear stress, respectively.

Johnson (1942) noticed that Manning's equation is valid only for rough-wall turbulent flow. For flume experiments with glass sidewalls, the corresponding flow is smooth-wall turbulent flow which is governed by Prandtl's friction law (Schlitchting 1979):

$$\frac{1}{\sqrt{f_w}} = 2\log_{10}\left(\mathsf{R}_w\sqrt{f_w}\right) - 0.8 \tag{3}$$

where f_w and $\mathsf{R}_w = 4R_wV/\nu$ are the sidewall friction factor and Reynolds number, with ν as kinematic water viscosity. In terms of f_w and f (the overall friction factor), Einstein's second hypothesis states:

$$V = \sqrt{\frac{8gR_wS_f}{f_w}} = \sqrt{\frac{8gRS_f}{f}} \tag{4}$$

where R is the overall hydraulic radius. Equation (4) results in:

$$R_w = \left(\frac{f_w}{f}\right)R, \ \ \mathsf{R}_w = \frac{4R_wV}{\nu} = \left(\frac{f_w}{f}\right)\mathsf{R} \tag{5}$$

where $\mathsf{R} = 4RV/\nu$ is the overall Reynolds number. Inserting R_w from Eq. (5) into Eq. (3) gives:

$$\frac{1}{\sqrt{f_w}} = 2\log_{10}\left(\frac{\mathsf{R}}{f}f_w^{3/2}\right) - 0.8 \tag{6}$$

which was solved iteratively for f_w previously. With ρ as water density, the sidewall shear stress τ_w then follows from:

$$\tau_w = \frac{f_w}{8}\rho V^2 \tag{7}$$

and the sidewall hydraulic radius R_w is found from Eq. (5). Finally, the bed shear stress τ_b is obtained from Eq. (2). Alternatively, τ_b is often found in terms of the bed friction factor:

$$f_b = f + \frac{2h}{b}(f - f_w) \tag{8}$$

resulting from the force balance: $(b+2h)\tau_o = b\tau_b + 2h\tau_w$ with τ_o as the overall boundary shear stress. This procedure is called Einstein-Johnson's sidewall correction validated with data from uniform and non-uniform, sub- and supercritical flows (Guo 2015); yet, it is seldom cited in literature because a simple explicit solution to Eq. (6) has not been found.

Practically, Vanoni and Brooks' (1957) procedure is used, which is just a graphical solution to Eq. (6) in terms of f_w versus R/f. For computer program, the graphical solution is further approximated analytically (Julien 1995; Cheng and Chua, 2005; Cheng *et al.* 2010; Cheng 2011). Unlike these approximations, this research solves Eq. (6) exactly in terms of the Lambert W-function; it also extends Einstein-Johnson's procedure to transitional- and rough-sidewall turbulent flows.

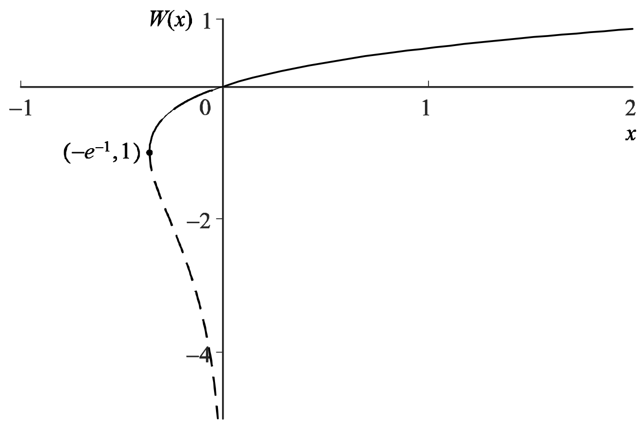

Figure 1: Plot of Lambert W-function

Exact Einstein-Johnson's Procedure

The Lambert W-function has been used by Sonnad and Goudar (2004) to solve the Colebrook (1939) equation for pipe friction factor. Similarly, this function can also solve Johnson's sidewall correction Eq. (6) explicitly. Mathematically, the Lambert W-function is defined by one of the following three equations:

$$We^W = x \tag{9}$$

$$\ln W + W = \ln x \tag{10}$$

$$W(x) = \ln \frac{x}{W(x)} = \ln \cfrac{x}{\ln \cfrac{x}{\ln \cfrac{x}{\ddots}}} \tag{11}$$

MatLab has this function named by "lamberw(x)"; Mathematica, Maple and Excel have it as a built-in function "LambertW(x)". The function $W(x)$ is plotted in Fig. 1, showing two branches: the upper one (solid line) is called the primary branch which is similar to the logarithmic function as $x \to \infty$; the lower branch is denoted by $W_{-1}(x)$. Only the primary branch is interesting in this research below.

Rearranging Eq. (6) in the form of Eq. (10) results in

$$\ln\left(\frac{\ln 10}{6}\frac{1}{\sqrt{f_w}}\right) + \frac{\ln 10}{6}\frac{1}{\sqrt{f_w}} = \ln x \tag{12}$$

with

$$x = \left(\frac{9}{400}\frac{\mathsf{R}}{f}\right)^{1/3} \tag{13}$$

Therefore, Eq. (12) has the exact solution:

$$\frac{\ln 10}{6}\frac{1}{\sqrt{f_w}} = W(x) = -\ln\frac{W(x)}{x} \tag{14}$$

where Eq. (11) is used. Equation (14) yields:

$$f_w = \frac{1}{36}\left[\log_{10}\frac{W(x)}{x}\right]^{-2} \tag{15}$$

The bed friction factor f_b then follows from Eq. (8). Clearly, Eq. (15) is explicit, exact and brief so that it makes Einstein-Johnson's sidewall correction procedure simple and accurate.

Generalized Einstein-Johnson's procedure

Equation (15) is limited to smooth-sidewall turbulent flow. If the sidewall flow is rough-wall turbulent flow, then the von Karman friction law (Schlitchting 1979), in terms of hydraulic radius, should be used:

$$\frac{1}{\sqrt{f_w}} = 2\log_{10}\frac{2R_w}{k_w} + 1.74 \tag{16}$$

where k_w is the equivalent Nikuradse sand roughness. Applying R_w from Eq. (5) to Eq. (16) and rearranging it according to Eq. (10) results in

$$\ln\left(\frac{\ln 10}{4}\frac{1}{\sqrt{f_w}}\right) + \frac{\ln 10}{4}\frac{1}{\sqrt{f_w}} = \ln y \tag{17}$$

with

$$y = \sqrt{\frac{\pi^2 R}{2fk_w}} \tag{18}$$

Equation (17) has the exact solution:

$$\frac{\ln 10}{4}\frac{1}{\sqrt{f_w}} = W(y) \tag{19}$$

or

$$f_w = \frac{1}{16}\left[\log_{10}\frac{W(y)}{y}\right]^{-2} \tag{20}$$

This equation is considered the extension of Einstein-Johnson's procedure for rough-sidewall turbulent flow.

For transitional-sidewall turbulent flow, the Colebrook (1939) equation is used, resulting in:

$$\frac{1}{\sqrt{f_w}} = -2\log_{10}\left(\frac{2.51f}{\mathsf{R}f_w^{3/2}} + \frac{1}{3.71f_w}\frac{fk_w}{4R}\right) \tag{21}$$

where Eq. (5) is used. This equation can be solved exactly by applying the generalized Lambert W-function (Scott *et al*. 2006), resulting in:

$$\frac{1}{\sqrt{f_w}} = -\frac{3}{2}\log_{10}\left[\left(\frac{W(x)}{x}\right)^4 + \left(\frac{W(y)}{y}\right)^{8/3}\right] \tag{22}$$

which is compared with the numerical solution of Eq. (21) in Fig. 2 showing perfect agreement. Therefore, Eq. (22) is the extension of Einstein-Johnson's procedure to the transitional-sidewall turbulent flow. Besides, (i) as $k_w = 0$ (smooth wall), $y \to \infty$; according to Eq. (9), one has $\lim_{y\to 0} W(y)/y = \lim_{y\to\infty}\exp(-W) = 0$ so that Eq. (22) reduces to Eq. (15); (ii) as $\mathsf{R}/f \to \infty$ (rough wall), $x \to \infty$; one has $\lim_{x\to\infty} W(x)/x = \lim_{x\to\infty}\exp(-W) = 0$ so that Eq. (22) reduces to Eq. (20). Briefly, Eq. (22) is the generalized Einstein-Johnson's sidewall correction equation, which is applicable for smooth-, transitional-, and rough-sidewall turbulent flows. Furthermore, this research makes Einstein-Johnson's sidewall correction as a two-step procedure: (i) calculate f_w from Eq. (22); and (ii) calculate f_b from Eq. (8). This procedure is applicable for flows in laboratory flumes, canals, streams, and rivers.

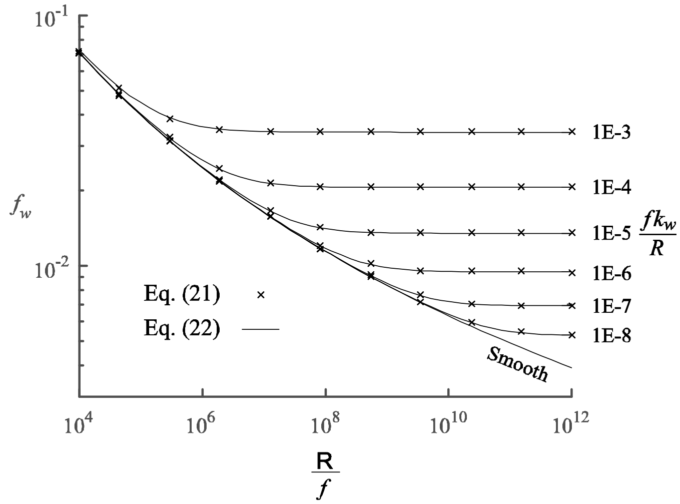

Figure 2: Comparison of Eq. (22) with Eq. (21)

Application Example

Einstein-Johnson's procedure has been systematically validated in Guo (2015) with data from uniform, non-uniform, sub-, and supercritical flows. This example only demonstrates the application of the exact procedure. Consider a laboratory experiment (Song 1994) in a rectangular flume 60 cm wide with hydraulically smooth sidewalls and an artificially roughened bottom which had a slope $S_o = 0.0025$. Given water temperature $T = 19.1\,^\circ\mathrm{C}$, discharge $Q = 0.09\,\mathrm{m}^3/\mathrm{s}$, and flow depth $h = 20.4\,\mathrm{cm}$, estimate the values of the bed friction factor f_b and the bed shear velocity u_{*b}.

For the overall flow, one can easily obtain: $V = 0.735\,\mathrm{m}/\mathrm{s}$, $R = 0.121\,\mathrm{m}$, $\mathrm{R} = 3.54 \times 10^5$, and $f = 0.0441$ (from Eq. 4). For the sidewall flow, the parameter x from Eq. (13) is

$$x = \left(\frac{9}{400}\frac{\mathrm{R}}{f}\right)^{1/3} = \left(\frac{9}{400}\frac{3.5361 \times 10^5}{0.0441}\right)^{1/3} = 56.5$$

The sidewall friction factor from Eq. (15) is then

$$f_w = \frac{1}{36}\left[\log_{10}\frac{W(x)}{x}\right]^{-2} = \frac{1}{36}\left[\log_{10}\frac{W(56.5)}{56.5}\right]^{-2} = 0.0169$$

which gives, from Eq. (8), the bed friction factor as:

$$f_b = f + \frac{2h}{b}(f - f_w) = 0.0441 + \frac{2(0.204)}{0.6}(0.0441 - 0.0169) = 0.0625$$

Therefore, the bed shear velocity is:

$$u_{*b} = V\sqrt{\frac{f_b}{8}} = (0.735)\sqrt{\frac{0.0625}{8}} = 0.0650\,\mathrm{m}/\mathrm{s} = 6.50\,\mathrm{cm}/\mathrm{s}$$

which is close to 6.38 cm / s from the log-law velocity distribution data, and 6.39 cm / s from Guo's (2015) method based on the cross-sectional velocity distribution.

If the sidewalls are concrete with roughness $k_w = 1\,\text{mm}$, the parameter y from Eq. (18) is also required:

$$y = \sqrt{\frac{\pi^2 R}{2 f k_w}} = \sqrt{\frac{\pi^2 (0.121)}{2 (0.0441) (10^{-3})}} = 116$$

The sidewall friction factor then follows from Eq. (22):

$$f_w = \frac{4}{9} \left\{ \log_{10} \left[\left(\frac{W(56.5)}{56.5} \right)^4 + \left(\frac{W(116)}{116} \right)^{8/3} \right] \right\}^{-2} = 0.0275$$

The bed friction factor is then:

$$f_b = 0.0441 + \frac{2 (0.204)}{0.6} (0.0441 - 0.0275) = 0.0554$$

and the bed shear velocity is:

$$u_{*b} = (0.735) \sqrt{\frac{0.0554}{8}} = 0.0612\,\text{m}/\text{s} = 6.12\,\text{cm}/\text{s}$$

which is smaller than that from the flume with smooth sidewalls. This result is reasonable because the overall shear stress τ_o does not change in the two cases.

Conclusions

This note presented an explicit and exact procedure for Einstein-Johnson's sidewall correction in open channel flow. It showed that Johnson's sidewall friction equation can be solved with the Lambert W-function that can be readily implemented with Matlab, Mathematica, Maple, Excel, and other software thereby facilitating the sidewall correction procedure for flume experiments on sediment transport, bridge scour, and vegetated flow. Besides, by applying the Colebrook friction law, a generalized Einstein-Johnson's procedure was proposed, which is applicable for smooth-, transitional- and rough-sidewall turbulent flows. A demonstrated example showed that the bed shear velocity from the proposed procedure agrees with the values from experiment and Guo's cross-sectional velocity distribution.

Notation

The following symbols are used in this technical note:

A_b, A_w	=	sub-areas for bed and sidewall, respectively (m^2);
b	=	channel width (m);
f	=	overall friction factor (–);
f_b	=	bed friction factor (–);
f_w	=	sidewall friction factor (–);
g	=	gravity acceleration (m s^{-2});

h	=	flow depth (m);
k_w	=	equivalent Nikuradse roughness for sidewall (m);
n_w	=	sidewall Manning's coefficient (s $m^{-1/3}$);
R	=	Reynolds number based on $4R$ and average velocity (–);
R_w	=	sidewall Reynolds number (–);
R	=	radius (m);
R_b	=	bed hydraulic radius (m);
R_w	=	sidewall hydraulic radius (m);
S_f	=	energy slope (–);
S_o	=	bottom slope (–);
V	=	cross-sectional average velocity (m^3 s^{-1});
$W(x)$	=	Lambert W-function (–);
x, y	=	independent parameters (–);
γ	=	specific water weight (N m^{-3});
ν	=	kinematic fluid viscosity (m^2 s^{-1});
ρ	=	density of fluid (kg m^{-3});
τ_o	=	overall boundary shear stress (Pa);
τ_b	=	bed shear stress (Pa); and
τ_w	=	sidewall shear stress (Pa).

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