Comparison of Current Methods for the Evaluation of Einstein's Integrals

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Abstract: Einstein's integrals constitute one of the salient developments in theoretical sediment mechanics. An analysis of the accuracy and computational efficiency of proposed methods for the calculation of the Einstein's integrals is presented. First, the accuracy of those techniques is determined using comparisons against highly accurate numerical results. For an infinite series solution, a study of accuracy versus number of terms in the partial sum is performed. Then, the central processing unit (CPU) times of the procedures are determined and compared over a full set of Rouse numbers and relative bedload-layer thicknesses. Finally, parallel versions of the methods are presented, and their parallel efficiency is assessed. Based on the criteria of accuracy, CPU time, and parallelization efficiency, it is concluded that the method by Guo and Julien, with modifications by Srivastava, is overall more efficient for implementation in sediment-transport codes. **DOI:** 10.1061/(ASCE)HY.1943-7900.0001240.

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Introduction

Numerous formulations have been proposed for the computation of the total sediment- transport load (Julien 2002; Parker 2004; Vanoni 2006; García 2008). Existing methods can be categorized into three main groups (Julien 2002): (1) formulas that are derived using regressions on experimental data; (2) formulas based on the balance of energy, such that the work done to carry particles is related to the energy expenditure; and (3) formulations based on other first principles. Among those procedures, Einstein's approach (Einstein 1950) is based on first principles, and is built on the rigorous foundation of continuum mechanics. Einstein's method is widely considered as one of the cornerstones of sediment mechanics (Julien 2002; Guo and Julien 2004; García 2008; Shah-Fairbank et al. 2011).

This method makes use of two integrals for the calculation of the suspended-sediment load. Without considering the multiplying factors, Einstein's first (J_1) and second (J_2) integrals are defined as

$$J_1(E,z) = \int_E^1 \left(\frac{1-y}{y}\right)^z dy \tag{1}$$

$$J_2(E,z) = \int_E^1 \left(\frac{1-y}{y}\right)^z \ln y\, dy \tag{2}$$

where E = relative bedload-layer thickness, $E \in [0.0001, 0.1]$ (usually considered as $E = 2d/H$ in the absence of bedforms/vegetation, in which d is the bed particle diameter and H indicates the water depth); y = dimensionless vertical coordinate; and z = Rouse number, defined as the particle fall velocity divided by the product of the von Kármán constant, κ, the inverse of the Schmidt number (β) (Bombardelli and Jha 2009; Jha and Bombardelli 2009, 2010), and the shear velocity, u^*. As most of the transport happens as bedload when the Rouse number is higher than 5 or 6, the Rouse number is assumed to vary between 0.1 and 6 in practical applications of the Einstein's method (Julien 2002; García 2008).

Analytical solutions of those integrals do not exist; therefore, Einstein (1950) provided nomograms for their calculation. Because using nomograms impedes the automation of Einstein's approach, several researchers developed simplifications, as is the case of Colby and Hubbell (1961), Toffaleti (1968), and Simons et al. (1981).

Einstein's method remained mostly unused in computer codes until very recently (Abad et al. 2008; Shah-Fairbank et al. 2011). In fact, the considerable computational effort associated with the evaluation of Einstein's integrals [as a result of the sharp gradients in the integrand functions near the bed (Nakato 1984; Zamani and Bombardelli 2016)] hinders the use of the method. To overcome this difficulty, several authors devised schemes to approximate the Einstein integrals, from computational approximations, to the use of convergent series solution, to regression-based schemes [Nakato (1984), Guo and Wood (1995), Guo and Julien (2004), Abad and García (Abad et al. 2006), Roland and Zanke (Abad et al. 2006), Srivastava (Abad et al. 2006), García (2008), Abad et al. (2008), and Shah-Fairbank et al. (2011)]. Given this menu of methods, there is the natural question as to which ones are the most convenient for each particular case.

In this paper, a systematic study of existing techniques of approximation of Einstein's integrals is presented, including the method by Nakato (1984), the series-based scheme by Guo and Julien (2004), the regression formula by Abad and García (Abad et al. 2006), the modification of Guo and Julien's approach by Roland and Zanke (Abad et al. 2006), and the method by Srivastava (Abad et al. 2006). The authors endeavor to uncover singularities or

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regions of inaccuracy in these methods, to provide *optimal* solutions in different ranges of all admissible Rouse numbers and relative bedload-layer thicknesses. The procedures are analyzed based on three criteria: (1) accuracy, (2) computational efficiency [i.e., central processing unit (CPU) time], and (3) the efficiency of parallel versions of those schemes.

Existing methods are briefly introduced in the next section. (Some details of their formulations are provided in Appendices I–IV.) Subsequently, the accuracy and efficiency of the techniques are assessed. Then, the authors present parallel versions of those approaches and assess their efficiency as well. An overall evaluation of all methodologies is provided by the end of the paper.

Methods for the Calculation of Einstein's Integrals

Evidently, the first effort to compute Einstein's integrals was conducted by Nakato (1984). Nakato divided the integrals into two zones of mild and sharp variations. Nakato computed the integrals in the former zone numerically, and devised an analytical solution for the latter (Appendix I).

The second approximation to Einstein's integrals was developed by Guo and Wood (1995). They recast a modification of the first Einstein integral into the beta function. They also used the first terms in the expansion of an integral, similar to the second Einstein integral, to approximate this second integral. Their derivation was only applicable to less-than-unity Rouse numbers; therefore, it is not considered in this paper.

The first major step toward the practical solution of Einstein's integrals was presented by Guo and Julien (2004). Guo and Julien resolved the issues of values of the Rouse number in the method by Guo and Wood (1995). In addition, they reworked both integrals into a recursive formula. They derived an infinite series–based solution for their recursive equation. An overview of Guo and Julien's method is provided in the Appendix II.

The paper by Guo and Julien (2004) prompted three discussions in 2006, by Abad and García, by Roland and Zanke, and by Srivastava, all compiled into a single note (Abad et al. 2006). Abad and García devised a regression-based polynomial approximation of the solution of each integral (Appendix III). Abad and García claimed that their regression formula is more practical and easy to implement in sediment-transport codes (Abad et al. 2008).

Roland and Zanke (Abad et al. 2006) built explicit expressions for the Einstein integrals (Appendix IV); however, their method has large discrepancies in the values of the J_2 integral for high relative bedload-layer thicknesses ($E > 0.01$) with respect to the *exact* value of the integrals. The suggested algorithm presents singularities in the integer Rouse numbers, as the authors themselves acknowledged [Figs. 1–3 in Roland and Zanke (Abad et al. 2006)]. Their study was the first work to discuss computational efficiency.

Finally, Srivastava, in Abad et al. (2006), conducted a rigorous mathematical study of convergence regions of the partial sums by Guo and Julien. Srivastava derived an explicit expression for the recursive formula (Appendix V). In addition, he proposed remedies for the problem of singularities in the series representation of the Einstein integrals.

Guo and Julien (Abad et al. 2006) responded to these three comments in a closure. They stated that Roland and Zanke's method is eventually equivalent to the combination of Guo and Wood's (1995) and Guo and Julien's (2004) algorithms. They conducted a study of the computational time of all methods and used them in a real-world example of sediment transport. Later, their algorithm was successfully implemented in sediment-transport codes and verified in applications (Shah-Fairbank et al. 2011).

In the next section, computational efficiency and accuracy of these methods are discussed.

Assessment of the Efficiency and Accuracy of Existing Methods

The authors started by developing a rigorous verification of all codes that they implemented in *MATLAB* for the previously mentioned methods. To that end, the authors used two high-order ordinary differential equation (ODE) solvers written in *MATLAB*: the NIntegrate command of *Wolfram Mathematica*, and stand-alone computations in Microsoft *Excel*. The results of the codes were compared against integral values in Table 1 of the closure by Guo and Julien (Abad et al. 2006) and against the values of the integral J_1 for half arguments from Guo and Julien (2004). Then, the authors evaluated the error for each method over a comprehensive data set of 960 pairs of values in the (E, z) space. The errors of the prediction of those five approaches are shown in Figs. 1 and 2. In those figures, and unless noted, *error* refers to the relative error of each scheme in which the results are compared against numerical values obtained with the composite Simpson method with 10,000 points (Appendix VI). For the J_1, Guo and Julien's method was used with 10 terms in the partial sum of Eq. (7). For the J_2, Guo and Julien's method was used with the Eq. (10) closure for the first infinite sum and 50 terms in the second partial sum. Fig. 3 shows the CPU time of the methods, with different parameters, to calculate Einstein's integrals for the same data set of (E, z). (The number after *Guo Julien* refers to the number of terms in the partial sum of the infinite series in J_2). Finally, statistical measures of accuracy are given in Table 1. Definitions of the statistical metrics are provided in Appendix VII.

Figs. 1 and 2 show that most methods provide an overall error smaller than 1% for most cases analyzed, which is acceptable in practice for sediment transport modeling. However, there are some areas in which the methods of Abad and García and Roland and Zanke are relatively inaccurate in J_1 and J_2 (not shown for Roland and Zanke's method in Fig. 2).

Nakato's procedure provides relatively low errors (with exceptions for high values of E and z), particularly for the J_2 integral. The accuracy of that method is comparable with that of the composite Simpson's integration with 1,000 points; nonetheless, Nakato's method is the slowest technique (Fig. 3). (Nakato's method was numerically integrated with 1,000 points in the slowly varying part.)

The method by Guo and Julien is the most accurate for approximation of the J_1 integral, even with 10-term truncation in the partial sum of the last term in the right side of Eq. (7); it is also the most accurate for J_2. The only issue with their algorithm is that its J_2 approximation slowly converges for large values of the relative bedload-layer thickness [see Srivastava in Abad et al. (2006)]. Guo and Julien's method is one of the fastest methods for computation of the J_1 integral (equal CPU time with Roland and Zanke's method). However, for the J_2 integral, this method requires almost an order of magnitude more time to provide results *with the equivalent accuracy of the Srivastava method*. In the test problems (Table 1), the authors found no significant improvement of the error metrics of computation of J_2 integral when using partial sums with more than 50 terms [first right-side term of Eq. (8)].

Additionally, the authors set up a test to evaluate the accuracy of Eq. (10) as an explicit closure for the first right-side partial sum in Eq. (8). Table 2 indicates that Guo and Julien's closure

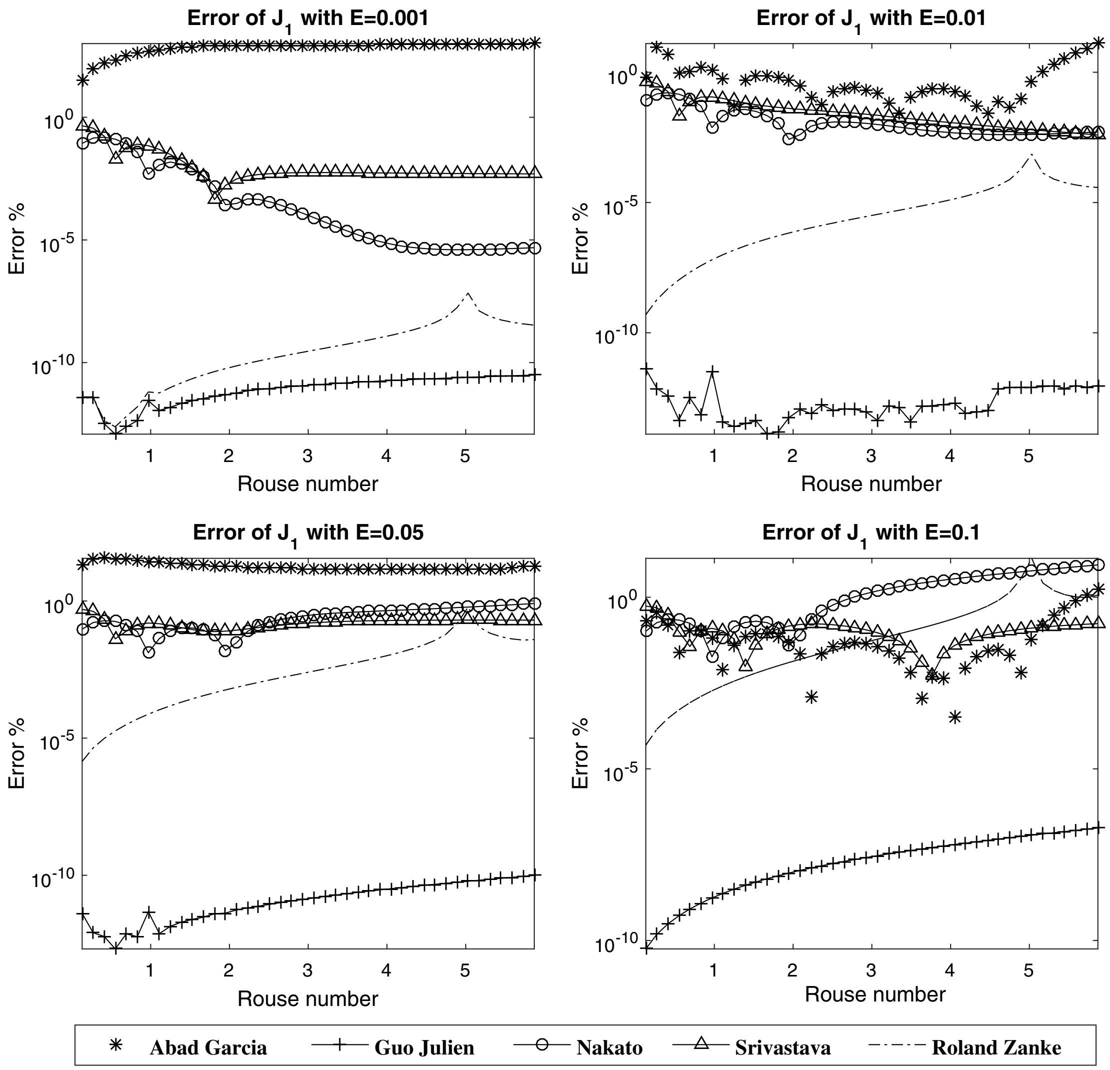

Fig. 1. Local relative error of the five methods for the calculation of the J_1

is effective—to the precision of less than 0.3%—for all ranges of the Rouse number (see line 1). In Table 2, the values indicate that the method of Guo and Julien without closure (lines 3 to 7) is relatively inaccurate in large Rouse numbers, and that at least 200 first terms are needed to keep the error below 1%. Furthermore, the explicit closure by Srivastava [Eq. (17)] is shown to be more accurate than the closure by Guo and Julien [Eq. (10)].

The Roland and Zanke's method has mostly moderate errors in J_1, but significant errors in J_2 as the relative bedload-layer reference height increases and the Rouse number is larger (not shown in this paper). This method is the fastest method for computing both integrals.

Srivastava's method has a rather low error in the prediction of the J_1 integral; the error smoothly reduces with the increase of Rouse number. For computing the J_2 integral, Srivastava's procedure is relatively accurate and is also among the fastest methods (in addition to Abad and García's method). A minor issue with Srivastava's scheme is that it has singularity near $z = 2.6$ [Fig. 2; see also Guo and Julien in Abad et al. (2006)].

Abad and García's regression has accuracy issues in several areas of the (E, z) plane. This method works better for higher relative bedload-layer thicknesses, and is fast and easy to implement.

In the analysis, these evaluations refer to the *numerical aspect* of uncertainty in modeling. It is well known that the uncertainty of any modeling activity can be calculated as follows (ASME 2009):

$$\delta = (\delta_{\text{model}} + \delta_{\text{numerical}} + \delta_{\text{input}}) - \delta_{\text{measurements}}$$

In other words, the total uncertainty in any simulation (δ) is the sum of the model structural uncertainty (δ_{model}), the uncertainty induced by numerical aspects of solving the equations of the model ($\delta_{\text{numerical}}$), and the uncertainty in the initial/boundary conditions and parameters (δ_{input}), minus the uncertainty due to the accuracy of the measurements ($\delta_{\text{measurements}}$). Thus, this research focuses on $\delta_{\text{numerical}}$ of the Einstein integral, and *does not* cover uncertainties in the Einstein method itself (model structural uncertainty), or uncertainties in the Rouse number and relative bedload-layer thickness (δ_{input}). In practical terms, there are severe concerns regarding the validity of the semilogarithmic velocity profile in the presence of large bedforms and vegetation, and of the Rousean profile itself (Julien 2002; García 2008; Bombardelli and Jha 2009); these correspond to δ_{model}. Furthermore, it is easily verifiable that errors of only 10% in E can lead to rather significant errors in the calculation of J_1 and J_2. Therefore, usually a 1% error is small enough in

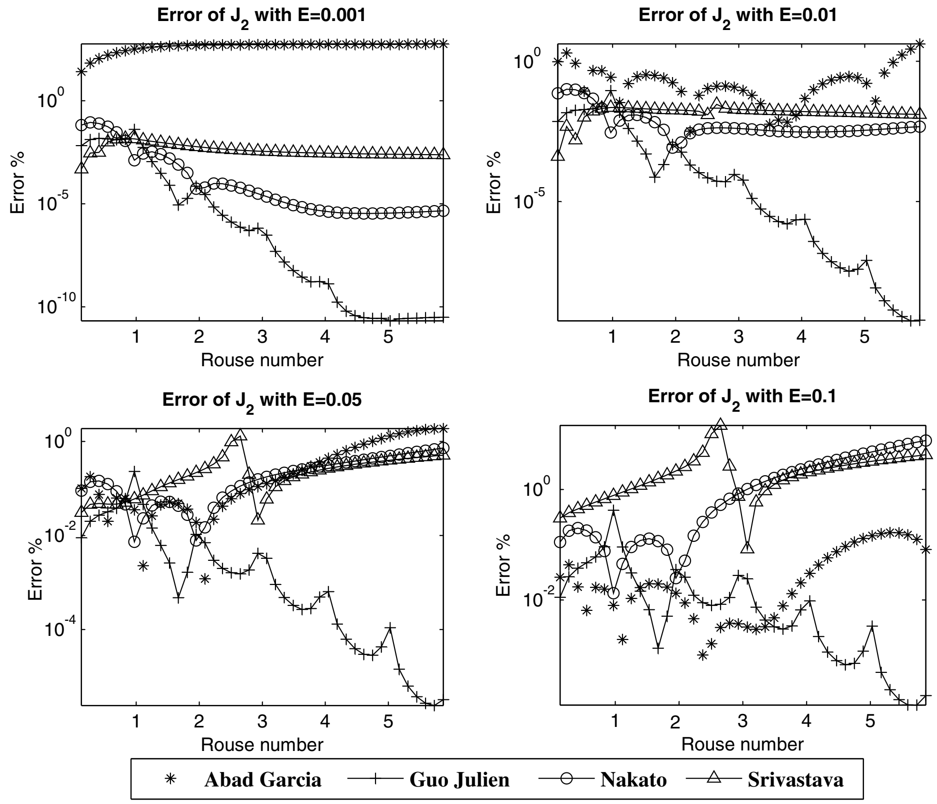

Fig. 2. Local relative error of the methods for the calculation of the J_2

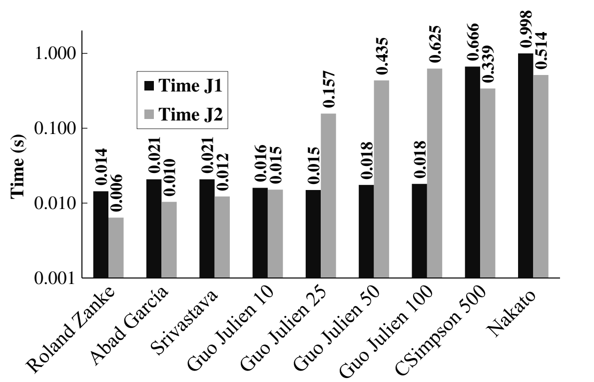

Fig. 3. CPU time of the methods for computing Einstein's integrals for 960 pairs of z and $\boldsymbol{E}$

sediment-transport models regarding the computation of Einstein's integrals. This paper provides the key to selecting the most adequate technique for each situation in which the input uncertainty has been reasonably estimated.

Parallelization Efficiency of Algorithms

Any advanced sediment-transport code requires simulation capability of multiple particle sizes, to mimic nonuniform distributions in natural streams (Papanicolaou et al. 2008). In sediment-transport software, hydrodynamics and transport solvers are commonly one-way coupled, assuming a dilute concentration of particles (Papanicolaou et al. 2008). Thus, all grain-size classes are transported by a unique flow field. These facts necessitate the use of parallel algorithms in sediment-transport solvers to increase the

Table 1. Global Accuracy of Existing Methods of Approximation of Einstein's Integrals

Measure	Abad and García	Guo and Julien ($N = 100$)	Nakato	Roland and Zanke	Srivastava	Composite Simpson ($N = 2{,}000$)
$\boldsymbol{J_1}$						
Bias	2.21E + 5	7.81E − 9	4.50E + 2	−1.24E + 1	−1.23E + 4	3.41E + 0
RMSE	5.93E + 6	4.50E − 7	2.93E + 3	4.25E + 1	1.28E + 5	6.37E + 1
Scatter index	1.70E + 0	1.38E − 13	8.99E − 4	1.30E − 5	3.94E − 2	1.95E − 5
$\boldsymbol{J_2}$						
Bias	−1.36E + 5	−6.03E + 4	−1.50E + 3	−3.38E + 10	1.10E + 3	−1.71E + 1
RMSE	1.12E + 7	7.11E + 5	1.13E + 4	7.16E + 11	2.46E + 4	3.21E + 2
Scatter index	−8.00E − 1	−5.09E − 2	−8.15E − 4	−2.12E + 1	−1.77E − 3	−2.31E − 5

Note: Accuracy was tested over a data set of 960 relative bedload-layer thicknesses and Rouse numbers.

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Table 2. Accuracy of the Partial Sum in the J_2 Series Approximations with Various Number of Terms

	Error[a] (%)				
Approximation	$z = 0.5$	$z = 1.5$	$z = 2.5$	$z = 3.5$	$z = 4.5$
Guo and Julien explicit closure [Eq. (10)] (1)	0.24	0.02	0.22	0.28	0.23
Srivastava explicit closure [Eq. (17)] (2)	0.05	<0.01	0.01	0.02	0.02
Partial sum with 10 terms[b] (3)	7.57	10.42	12.7	14.62	16.29
Partial sum with 20 terms[b] (4)	3.93	5.51	6.85	8.02	9.07
Partial sum with 50 terms[b] (5)	1.61	2.29	2.87	3.41	3.90
Partial sum with 100 terms[b] (6)	0.81	1.16	1.46	1.74	2.00
Partial sum with 200 terms[b] (7)	0.41	0.58	0.74	0.88	1.01

[a]Error = |"Exact" Value − Approximated Value/"Exact" Value| × 100.
[b]Number of terms in the partial sum on the first right-side term in Eq. (8).

Table 3. Speedup for the Parallelization of Existing Methods of Computation of Einstein's Integrals

Method	Runtime with single core (s)	Speedup with cores (s/s) 2	3	4
J_1				
Composite Simpson ($N = 2{,}000$)	4.260	1.84	3.03	3.32
Nakato	2.670	2.50	3.33	3.64
Guo and Julien[a]	0.890	1.60	1.83	1.88
J_2				
Composite Simpson ($N = 2{,}000$)	5.230	1.84	2.64	3.40
Nakato	1.753	1.72	2.46	2.96
Guo and Julien[a]	5.590	1.86	2.38	2.90

Note: Data set of 9,000 (E, z).
[a]This method was executed using Eq. (10) closure and the first 100 terms in the partial sum of Eq. (8).

computational efficiency (e.g., Keshtpoor et al. 2015). The integration of J_1 and J_2 was performed on multicore processors.

Parallel for Loops (*parfor*) of *MATLAB*'s Parallel Computing Toolbox was used for shared-memory parallelization of the calculations on multicore (MathWorks 2015). The performance of the parallelized versions was evaluated on an Intel i7-2670QM multicore processor using a data set of 9,000 pairs of inputs. The resulting speedups (Appendix VII) are given in Table 3.

The best performance is achieved by the composite Simpson's method, in which the speedup is close to the ideal line for parallel computing of Einstein's integrals. Nakato's procedure is slightly less efficient than the composite Simpson's method in the J_1 integral, but its speedup ratio is nearly linear and again close to the ideal line. In parallel computing of the J_2 integral, Nakato's method performs well for 2 and 3 cores; however, the linear upward speedup trend reaches a plateau for 4 cores. The third best speedup for the J_1 integrals is Guo and Julien's method (Table 3). This method for the J_2 integrals—with 100 terms in the partial sum—has a better speedup factor than Nakato's. Other techniques do not provide improvements in the use of parfor because of their inherent structure.

More work would be needed in this area to have unequivocal conclusions regarding parallel implementation of the methods.

Final Remarks and Conclusions

Five existing methods for the calculation of Einstein's integrals were compared in this paper. Error, CPU time, and the performance of their parallel implementation were evaluated. Sediment-transport modelers can use this information to select the most convenient method for computation of Einstein's integrals, setting a desired level of accuracy (usually 1%) and making their decision based on the computational time and range of parameters.

Considering the tradeoff between accuracy and computational time, the authors recommend the series-based-solution method by Guo and Julien for computing the J_1 integral with only 10 first terms, which is relatively fast and accurate. Guo and Julien's method shows superiority for parallel computing of the J_1 integral. In addition, Roland and Zanke's method is a reasonable one, and it is nearly an order of magnitude faster than Guo and Julien's method.

For sequential computing of the J_2 integral, the authors recommend Srivastava's modification to the Guo and Julien's method. It is accurate and faster than all other methods (except Abad and García's and Roland and Zanke's methods). In addition, the formula of Abad and García provides relatively accurate results for the J_2 integral in high relative bedload-layer thicknesses.

Table 4. Regression Coefficients for Eqs. (11) and (12)

E	C_0/D_0	C_1/D_1	C_2/D_2	C_3/D_3	C_4/D_4	C_5/D_5	C_6/C_6
0.001	8.0321	−26.273	−114.69	501.43	−229.51	41.94	−2.7722
	2.5779	−12.418	47.353	17.639	−13.554	2.8392	−0.2003
0.005	2.1142	−3.4502	12.491	60.345	−29.421	5.4215	−0.3577
	1.2623	1.0330	13.543	0.7655	−1.6646	0.3803	−0.0275
0.01	1.4852	0.2025	14.087	20.918	−10.91	2.034	−0.1345
	1.1510	2.1787	7.6572	−0.2777	−0.570	0.1424	−0.0105
0.05	1.1038	2.6626	5.6497	0.3822	−0.6174	0.1315	−0.0091
	1.2574	2.3159	1.9239	−0.3558	0.0075	0.0064	−0.0006
0.1	1.1266	2.6239	3.0838	−0.3636	−0.0734	0.0246	−0.0019
	1.4952	2.2041	1.0552	−0.2372	0.0265	−0.0008	−0.00005

Appendix I. Nakato's Method

Nakato (1984) separated both Einstein's integrals into two regions: near the relative bedload-reference level ($E < y < \epsilon$), and the upper region ($\epsilon < y < 1$), as follows:

$$J_1 = \int_E^1 \left(\frac{1-y}{y}\right)^z dy = \int_E^\epsilon \left(\frac{1-y}{y}\right)^z dy + \int_\epsilon^1 \left(\frac{1-y}{y}\right)^z dy \tag{3}$$

$$J_2 = \int_E^1 \left(\frac{1-y}{y}\right)^z \ln y\, dy = \int_E^\epsilon \left(\frac{1-y}{y}\right)^z \ln y\, dy + \int_\epsilon^1 \left(\frac{1-y}{y}\right)^z \ln y\, dy \tag{4}$$

He further integrated the upper region numerically with Simpson's rule, and derived the following formulas for the part close to the relative bedload-layer thickness:

$$\int_E^\epsilon \left(\frac{1-y}{y}\right)^z dy = F_1 + F_2 + F_3 \tag{5a}$$

in which F_i are defined as

$$F_1 = \frac{1}{1-z}(\epsilon^{1-z} - E^{1-z}); \qquad F_2 = \frac{z}{z-2}(\epsilon^{2-z} - E^{2-z});$$
$$F_3 = \frac{z(z-1)}{2(3-z)}(\epsilon^{3-z} - E^{3-z}) \tag{5b}$$

In the singularities at $z = 1$, $z = 2$, and $z = 3$, the following expressions can be used instead:

$$F_1 = \ln\frac{\epsilon}{E}; \qquad F_2 = -2\ln\frac{\epsilon}{E}; \qquad F_3 = 3\ln\frac{\epsilon}{E} \tag{5c}$$

$$\text{In turn, } \int_E^\epsilon \left(\frac{1-y}{y}\right)^z \ln y\, dy = G_1 + G_2 + G_3 \tag{6a}$$

in which the G_i are defined as

$$G_1 = \frac{\epsilon^{1-z}}{1-z}\left(\ln\epsilon - \frac{1}{1-z}\right) - \frac{E^{1-z}}{1-z}\left(\ln E - \frac{1}{1-z}\right);$$
$$G_2 = \frac{z\epsilon^{2-z}}{z-2}\left(\ln\epsilon - \frac{1}{2-z}\right) - \frac{zE^{2-z}}{z-2}\left(\ln E - \frac{1}{2-z}\right);$$
$$G_3 = \frac{z(z-1)\epsilon^{3-z}}{2(3-z)}\left(\ln\epsilon - \frac{1}{3-z}\right) - \frac{z(z-1)E^{3-z}}{2(3-z)}\left(\ln E - \frac{1}{3-z}\right) \tag{6b}$$

In the singularities at $z = 1$, $z = 2$, and $z = 3$, the following expressions can be used instead:

$$G_1 = \frac{1}{2}[(\ln\epsilon)^2 - (\ln E)^2]; \qquad G_2 = -(\ln\epsilon)^2 + (\ln E)^2;$$
$$G_3 = \frac{3}{2}[(\ln\epsilon)^2 - (\ln E)^2] \tag{6c}$$

Appendix II. Guo and Julien's Method

Guo and Julien (2004) derived closed-form, analytical solutions of the problem for integer values of the Rouse number; for noninteger values they derived the following formulas:

$$J_1(E,z) = \frac{z\pi}{\sin(z\pi)} - \Phi(z) \tag{7}$$

$$J_2(E,z) = \frac{z\pi}{\sin(z\pi)}\left[\pi\cot(z\pi) - 1 - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right)\right] - \left[\Phi(z)\left(\ln E + \frac{1}{z-1}\right) + z\sum_{n=1}^{\infty}\frac{(-1)^n}{(z-n)}\frac{\Phi(z-n)}{(z-n-1)}\right] \tag{8}$$

where $\Phi(z)$ is defined as

$$\Phi(z) = \frac{(1-E)^z}{E^{z-1}} - z\sum_{n=1}^{\infty}\frac{(-1)^n}{n-z}\left(\frac{E}{1-E}\right)^{n-z} \tag{9}$$

Guo and Julien suggested the following closure for the first infinite series in Eq. (8):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) \approx \frac{\pi^2}{6}\frac{z}{(1+z)^{0.7162}} \tag{10}$$

Appendix III. Abad and García's Regression

Abad and García (Abad et al. 2006) suggested the following formulas for the Einstein integrals:

$$J_1 = (C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + C_5 z^5 + C_6 z^6)^{-1} \tag{11}$$

$$J_2 = (D_0 + D_1 z + D_2 z^2 + D_3 z^3 + D_4 z^4 + D_5 z^5 + D_6 z^6)^{-1} \tag{12}$$

where the coefficients of Eqs. (11) and (12) are given in Table 4.

Appendix IV. Roland and Zanke's Method

Roland and Zanke (Abad et al. 2006) proposed the following expressions for the Einstein integrals:

$$J_1(E,z) = \left(\frac{1}{z-1}\right)\left[\frac{(1-E)^z}{E^{z-1}}\right] - \left(\frac{z}{z-1}\right) \times \left[\left(\frac{1}{z-2}\right)\left[\frac{(1-E)^{z-1}}{E^{z-2}}\right] - \left(\frac{z-1}{z-2}\right) \times \left(\left(\frac{1}{z-3}\right)\left[\frac{(1-E)^{z-2}}{E^{z-3}}\right] - \left(\frac{z-2}{z-3}\right) \times \left\{\frac{(z-3)\pi}{\sin[(z-3)\pi]} - \frac{E^{4-z}}{4-z}\right\}\right)\right] \tag{13}$$

$$J_2(E,z) = \left(\frac{1}{z-1}\right)\left(\ln E\frac{(1-E)^z}{E^{z-1}} - z\left\{\left(\frac{1}{z-2}\right) \times \left[\ln E\frac{(1-E)^{z-1}}{E^{z-2}} - (z-1)J_2(E,z-3)J_1(E,z-2)\right]\right\} + J_1(E,z)\right) \tag{14}$$

They also suggested the following expressions to approximate $J_2(E,z-3)$ and $J_1(E,z-2)$:

$$J_2(E, z-3) = -\frac{(z-2)\pi\psi(z)}{\sin[(z-2)\pi]} - \frac{E^{3-z}}{3-z}\ln E + \frac{E^{3-z}}{(3-z)^2} \quad (15a)$$

$$\psi(z) = (1-\gamma) - \ln|4-z| + \frac{1}{3-z} + \frac{1}{2(4-z)} + \frac{1}{24(4-z)^2} \quad (15b)$$

where γ = Euler-Mascheroni constant; and

$$J_1(E, z-2) = \left(\frac{1}{z-2}\right)\left[\frac{(1-E)^{z-1}}{E^{z-2}}\right] - \left(\frac{z-1}{z-2}\right)\left[\frac{(z-2)\pi}{\sin[(z-2)\pi]} - \frac{E^{3-z}}{3-z}\right] \quad (16)$$

Appendix V. Srivastava's Method

Srivastava (Abad et al. 2006) first suggested a more accurate explicit closure that replaces Eq. (10) by Guo and Julien (2004) with

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) \approx \ln(1+1.781z) - \frac{0.1361z}{(1+1.284z)^{2.15}} \quad (17)$$

Srivastava introduced a change of variable as $E_* = E/(1-E)$, and derived the following closed-form formulas for Einstein's integrals:

$$J_1(E_*, z) = -\frac{E_*^{1-z} - 1}{1-z} + 2.061\frac{E_*^{2-z} - 1}{2-z} - 1.385\frac{E_*^{2.6-z} - 1}{2.6-z} + \frac{0.3327}{0.6703+z} \quad (18)$$

$$J_2(E_*, z) = \frac{E_*^{1-z}[1-(1-z)\ln E_*] - 1}{(1-z)^2} - 1.903\frac{E_*^{2-z}[1-(2-z)\ln E_*] - 1}{(2-z)^2} + 2.022\frac{E_*^{2.6-z}[1-(2.6-z)\ln E_*] - 1}{(2.6-z)^2} - \frac{0.2914}{1.652+z} \quad (19)$$

Appendix VI. Composite Simpson Rule

The composite Simpson rule for numerical integration is provided below for n subintervals. This method has a truncation error of $O(h^4)$ (Press et al. 1992):

$$\int_a^b f(x)dx = \frac{h}{3}\sum_{j=1}^{\frac{n}{2}}[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] + \frac{h^4(b-a)}{180}\max|f^{(4)}(\mu)| \quad (20)$$

where $\mu \in [a, b]$; $h = (b-a)/n$; $x_0 = a$; $x_n = b$; and $x_j = a + jh$.

Appendix VII. Statistics of Model Skill Assessment

The following statistics are used to evaluate the differences among results of the methods, denoted by M, and values of a benchmark, indicated by B (Zamani and Bombardelli 2014):

1. $$\text{Bias} = \frac{1}{N}\sum_{i=1}^{N}(M_i - B_i) \quad (21)$$

   Bias is a measure of over- or underprediction; essentially a bias close to zero is ideal.

2. $$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(M_i - B_i)^2} \quad (22)$$

   Root mean square error (RMSE) is a metric for modeling error, which amplifies large errors over the computation domain.

3. $$\text{SI} = \frac{\text{RMSE}}{\frac{1}{N}\sum_{i=1}^{N} B_i} \quad (23)$$

   Scatter index (SI) is another measure of error, in which the RMSE is nondimensionalized by the average value of the benchmark. The SI is more informative than the RMSE, as high (or low) values of RMSE can be misleading in cases of extremely high (or low) values of model results.

4. Parallelization speedup is a metric in the evaluation of parallel computing efficiency that shows relative performance improvement as a task is executed on multiprocessors compared with a single processor. Speedup is the ratio of the time the computation of one processor divided by the time of computation with all processors.

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Discussion of "Comparison of Current Methods for the Evaluation of Einstein's Integrals" by Kaveh Zamani, Fabián A. Bombardelli, and Babak Kamrani-Moghaddam

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Introduction

The authors are appreciated for comparing the five existing methods for the calculation of Einstein's integrals. The authors have presented an analysis for the accuracy and computational speed of these five methods. The discussers would like to mention the following points.

The integration of Einstein's integrals results in hypergeometric functions that are not suitable for practical purposes due to their complexity.

By defining $E_* = E/(1-E)$, Srivastava (2006) truncated the series solutions of Einstein's integrals and derived the following regression-based approximations:

$$J_1(E,z) = \int_E^1 \left(\frac{1-y}{y}\right)^z dy \cong -\frac{E_*^{1-z}-1}{1-z} + 2.061\frac{E_*^{2-z}-1}{2-z} - 1.385\frac{E_*^{2.6-z}-1}{2.6-z} + \frac{0.3327}{0.6703+z} \tag{1}$$

$$J_2(E,z) = \int_E^1 \left(\frac{1-y}{y}\right)^z \ln(y)dy \cong \frac{E_*^{1-z}[1-(1-z)\ln(E_*)]-1}{(1-z)^2} - 1.903\frac{E_*^{2-z}[1-(2-z)\ln(E_*)]-1}{(2-z)^2} + 2.022\frac{E_*^{2.6-z}[1-(2.6-z)\ln(E_*)]-1}{(2.6-z)^2} - \frac{0.2914}{1.652+z} \tag{2}$$

where E = relative bed-load-layer thickness; and z = Rouse number. These approximations have singularities at $z = 1$, 2, and 2.6, which could simply be removed by considering $z = 1.001$, 2.001, and 2.601 ($\Delta z = 0.001$) instead of the previous values. Similarly, in the unlikely event of z being exactly equal to 1, 2, or 2.6, some limitations could be put into practice as indicated by Srivastava (2006)

$$\lim_{p\to 0}\frac{x^p-1}{p} = \ln(x) \tag{3}$$

$$\lim_{p\to 0}\frac{x^p[1-p\ln(x)]-1}{p^2} = -0.5[\ln(x)]^2 \tag{4}$$

The percentage errors [$100 \times (1 - \text{approximated value}/\text{exact numerical value})$] occurring in Eqs. (1) and (2) are depicted in Fig. 1. A perusal of Fig. 1 reveals that the approximate solution for J_2 [Eq. (2)] is sufficiently accurate for all practical purposes. The maximum percentage error of this approximation is less than 0.032% for all practical ranges of $z \in [0.1, 8]$ and $E \in [0.0001, 0.1]$. For computing the J_2 integral, Srivastava's approximation is very accurate with respect to the accurate numerical values of the integral even for high relative bed-load-layer thicknesses ($E = 0.1$). However, the percentage error involved in Eq. (2) is incorrectly shown in Fig. 2 of the original paper. Based on this figure, for $E = 0.05$ and 0.1, the percentage error of Eq. (2) exceeds 1%, which is unacceptable. Guo and Julien (2006) also incorrectly reported the percentage error involved in use of Eq. (2) was used. They reported a value of −28.742 (a 17% error) for J_2 at $z = 2.55$ and $E = 0.1$. Using Eq. (2) for $z = 2.55$ and $E = 0.1$ results in $J_2 = -24.5165$, which is comparable with the exact numerical value $J_2 = -24.5130$ (only 0.014% error). Moreover, Guo and Julien (2006) incorrectly reported high error for Eq. (2) at singular point $z = 2.6$ ($E = 0.1$). Using Eq. (2) for $z = 2.599$ and 2.601 ($\Delta z = \pm 0.001$), one respectively gets $J_2 = -26.7679$ and -26.8642, which are

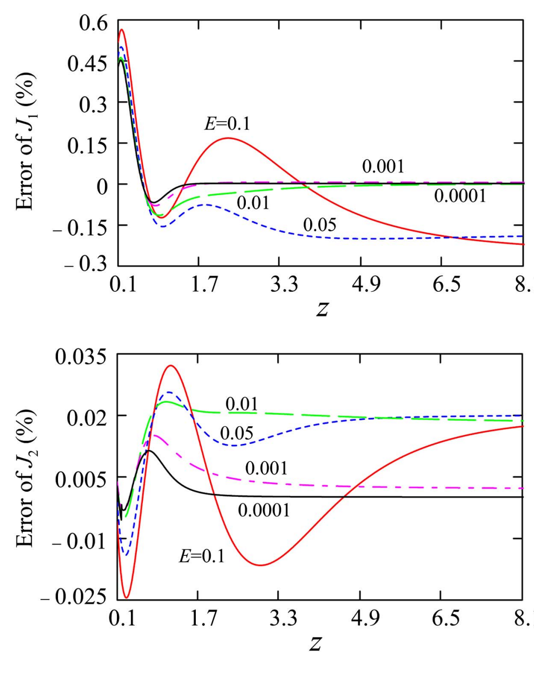

Fig. 1. Percentage errors in Eqs. (1) and (2) versus the Rouse number, z, for various relative bed-load-layer thicknesses E.

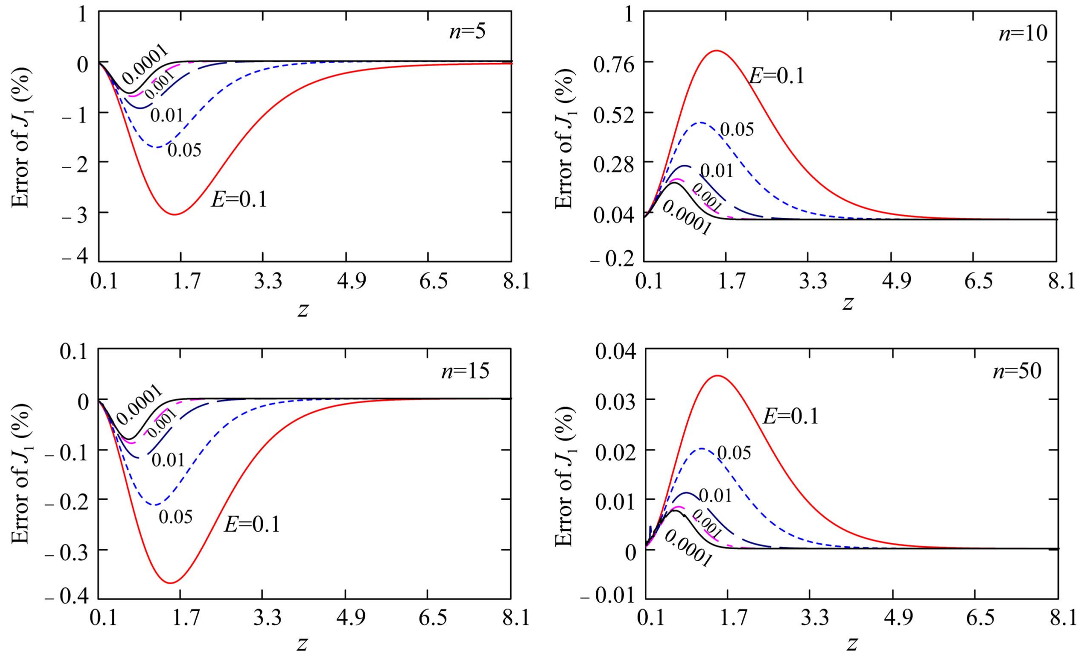

Fig. 2. Percentage error of infinite series Eq. (7) versus the Rouse number, z, for various numbers of terms.

completely comparable with the exact numerical value of $J_2 = -26.8121$ (only 0.015% error).

While the simple Srivastava approximation for J_2 [Eq. (2)] is very accurate for all practical purposes, the percentage error of Srivastava's approximation for J_1 [Eq. (1)] reaches up to 0.57%.

In the following sections, a new series solution for J_1 is presented, and in the next step the solution is truncated and new regression-based approximations for J_1 of the Einstein's integral are derived.

New Series Solution for J_1

Many sums of reciprocal powers could be expressed in terms of Lerch transcendent. The Lerch transcendent, ϕ, is given by

$$\phi(x, s, \alpha) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\alpha)^s} \tag{5}$$

In order to derive a new series of solutions for J_1, this integral is written as (Srivastava 2006)

$$\begin{aligned} J_1(E, z) &= \int_E^1 \left(\frac{1-y}{y}\right)^z dy \\ &= \int_{E_*}^1 x^{-z}(1+x)^{-2} dx + \int_0^1 x^z (1+x)^{-2} dx \end{aligned} \tag{6}$$

in which $E_* = E/(1-E)$.

Eq. (6) could be expressed in terms of $\phi(-E_*, 1, -z)$ and $\phi(-1, 1, z)$. The equation finally could be shown as

$$J_1(E, z) = 1 - \frac{E_*^{1-z}}{1+E_*} + z \sum_{n=1}^{\infty} (-1)^n \left(\frac{E_*^{n-z} - 1}{n-z} + \frac{1}{n+z}\right) \tag{7}$$

There is no sine function in Eq. (7). The computation of sine function (also logarithm function) in many computer languages is based on series expansions (Taylor series), which requires several terms of the argument to be computed and summed with each other. Some existing techniques of approximation of Einstein's integral J_1 have been expressed in terms of a sine function. Computational speed of these approximations could be limited due to the calculation of the sine function.

The percentage error of infinite series Eq. (7) versus a number of terms in the partial sum is shown in Fig. 2. The number of terms required varies according to the desired precision and increases for high relative bed-load-layer thicknesses ($E = 0.1$). For a maximum error less than 1%, at least nine first terms in the partial sum are required.

Truncated Regression-Based Approximations for J_1

To avoid the slow convergence, the infinite series Eq. (7) could be truncated to five first terms in the partial sum and the sixth term could be multiplied by a regression coefficient of 0.591 as

$$\begin{aligned} J_1(E, z) \cong{}& 1 - \frac{E_*^{1-z}}{1+E_*} + z \sum_{n=1}^{5} (-1)^n \left(\frac{E_*^{n-z} - 1}{n-z} + \frac{1}{n+z}\right) \\ &+ 0.591 z \left(\frac{E_*^{6-z} - 1}{6-z} + \frac{1}{6+z}\right) \end{aligned} \tag{8}$$

The percentage error of this truncated series solution is shown in Fig. 3. The maximum percentage error of this approximation is less than 0.036% for all practical ranges of $z \in [0.1, 8]$ and $E \in [0.0001, 0.1]$.

Considering the trade-off between accuracy and computational speed, the following approximation for J_1 was also developed based on the infinite series Eq. (7) and the curve fitting technique:

$$\begin{aligned} J_1(E, z) \cong{}& 1 - \frac{E_*^{1-z}}{1+E_*} - z\frac{E_*^{1-z} - 1}{1-z} + 1.017 z \frac{E_*^{2-z} - 1}{2-z} \\ &- 0.595 z \frac{E_*^{2.74-z} - 1}{2.74-z} - \frac{0.6z}{0.84+z} \end{aligned} \tag{9}$$

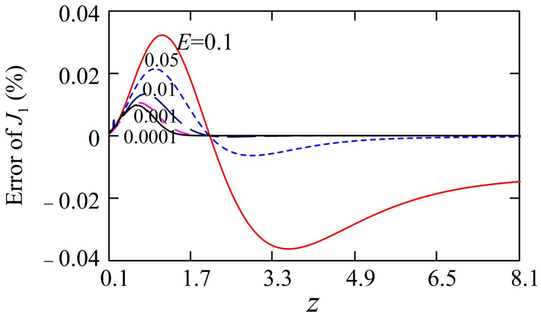

Fig. 3. Percentage error of truncated series Eq. (8) versus the Rouse number, z, for various relative bed-load-layer thicknesses E.

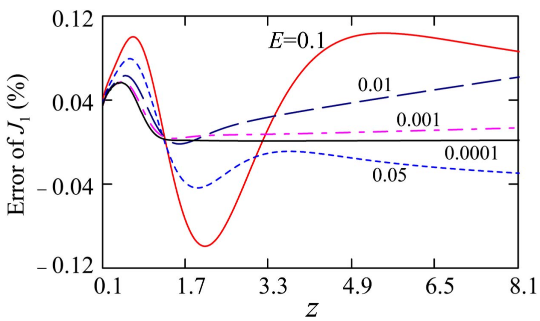

Fig. 4. Percentage error of simple truncated series Eq. (9) versus the Rouse number, z, for various relative bed-load-layer thicknesses E.

The percentage error of this truncated regression-based series solution is shown in Fig. 4. The maximum percentage error of this approximation is less than 0.1% for all ranges of z and E encountered in practical applications.

The accuracy is not the only criterion to select an approximation solution, especially for computer application. The approximations proposed herein for J_1 represent different trade-offs between mathematical complexity (computational speed) and accuracy. Obviously, an increased accuracy could be obtained at the price of augmented computational costs. It seems that the proposed simple approximation in Eq. (9) for J_1 and the proposed simple approximation in Eq. (2) by Srivastava (2006) for J_1 are preferred compared with other existing approximations in terms of both accuracy and computational speed.

Partial Sum in J_1 Series Solution by Guo and Julien

Guo and Julien (2004) suggested the following closure for using in their infinite series solution of J_1:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \cong \frac{\pi^2 z}{6(1+z)^{0.7162}} \tag{10}$$

The following approximation was also derived by Srivastava (2006) with the use of the limiting behavior over the entire range $z > 0$:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \cong \ln(1+1.781z) - \frac{0.1361z}{(1+1.284z)^{2.150}} \tag{11}$$

Eq. (11), which includes the logarithm function, allows a little increase in accuracy compared with Eq. (10), but at the cost of lower computational speed.

In this discussion, a more accurate explicit function is proposed as follows:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \cong \frac{1.672z}{(1.035+z)^{0.73}} \tag{12}$$

The maximum percentage error of Eq. (12) is less than 0.3%, while the maximum percentage errors of Eqs. (10) and (11) are about 2% for practical ranges of $z \in [0.1, 8]$.

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Closure to "Comparison of Current Methods for the Evaluation of Einstein's Integrals" by Kaveh Zamani, Fabián A. Bombardelli, and Babak Kamrani-Moghaddam

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The writers would like to start by sincerely thanking the discussers for their interest in our work. In the writers' opinion, the original paper, the discussion, and this closure all show that the subject of Einstein's integrals is still very important, and that it also offers opportunities for sound research. The writers would be very interested in seeing more efficient methods for the evaluation of these integrals in the near future.

Before presenting responses to the discussers' commentary, the writerss would like to mention a crucial concept that becomes a key component in any computational modeling activity: reproducibility (Leveque 2013; Stodden et al. 2013; Hutton et al. 2016). In the case of the original paper, even though the schemes and implementation were relatively simple, all subroutines were stored as free software in an online repository for anybody to check the work. In the numerical solution of any complex hydrological phenomenon—and in general in any modeling activity—special care needs to be taken into consideration to avoid common mistakes. Specifically, it becomes mandatory to verify and validate the codes. The writers believe that respecting the basics of transparency and reproducibility in modeling prevents claims that cannot be supported by computational results. The authors provided an extensive verification procedure, which used as elements the seminal work by Einstein (1950), the original paper by Guo and Julien (2004), and Table 1 in the closure by Guo and Julien (2006). The authors also checked their scripts with using diverse languages.

In their work, the discussers state the following: "Guo and Julien (2006) also incorrectly reported the percentage error involved because Eq. (2) was used. They reported a value of −28.742 (a 17% error) for J_2 at $z = 2.55$ and $E = 0.1$. Using Eq. (2) for $z = 2.55$ and $E = 0.1$ results in $J_2 = -24.5165$, which is comparable with the exact numerical value $J_2 = -24.5130$ (only 0.014% error)." The discussers are right in this statement; the values in the line corresponding to Srivastava's method in Table 1 of the original closure are incorrect. The writers reran the four cases of the J_2 integral included by the discussers with the correct implementation of Srivastava's method, and compared them with the results of a cubic Simpson method with 20,000 points (Fig. 1). In the writers' correct implementation of Srivastava's method, the switch to the expression for singularity was set to $\Delta z < 0.01$. (It is difficult to compare the writers' results with those of the discussers because the discussers have not explained how they computed the exact value of the integrals.)

After the "Introduction," the discussers used Srivastava's manipulation of the J_1 integral and derived new explicit formulas for the calculation of J_1. Their derivation was based on a modified power series by which they introduced two truncated series [Eqs. (8) and (9)] to account for the original infinite series [Eq. (7)]. Indeed, it is refreshing to see new methods for Einstein's integrals in line with what was stated at the beginning of this closure; however, the issue is whether these methods are accurate and fast enough as compared with alternatives.

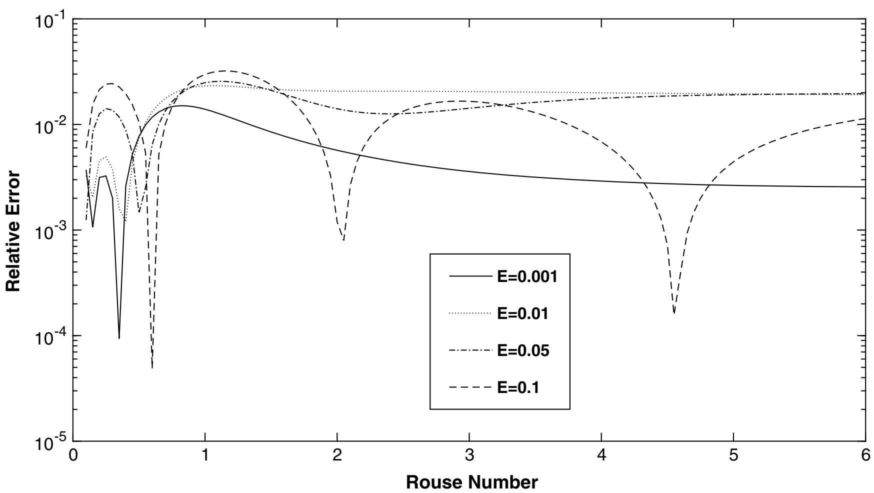

Fig. 1. Relative error (%) of Srivastava method for calculation of J_2 integral with the switch of $\Delta z = 0.01$ near the singularities for various bed-load-layer thicknesses and Rouse numbers. Herein, relative error is defined as Error = |Exact value − Approximated value/Exact value| × 100.

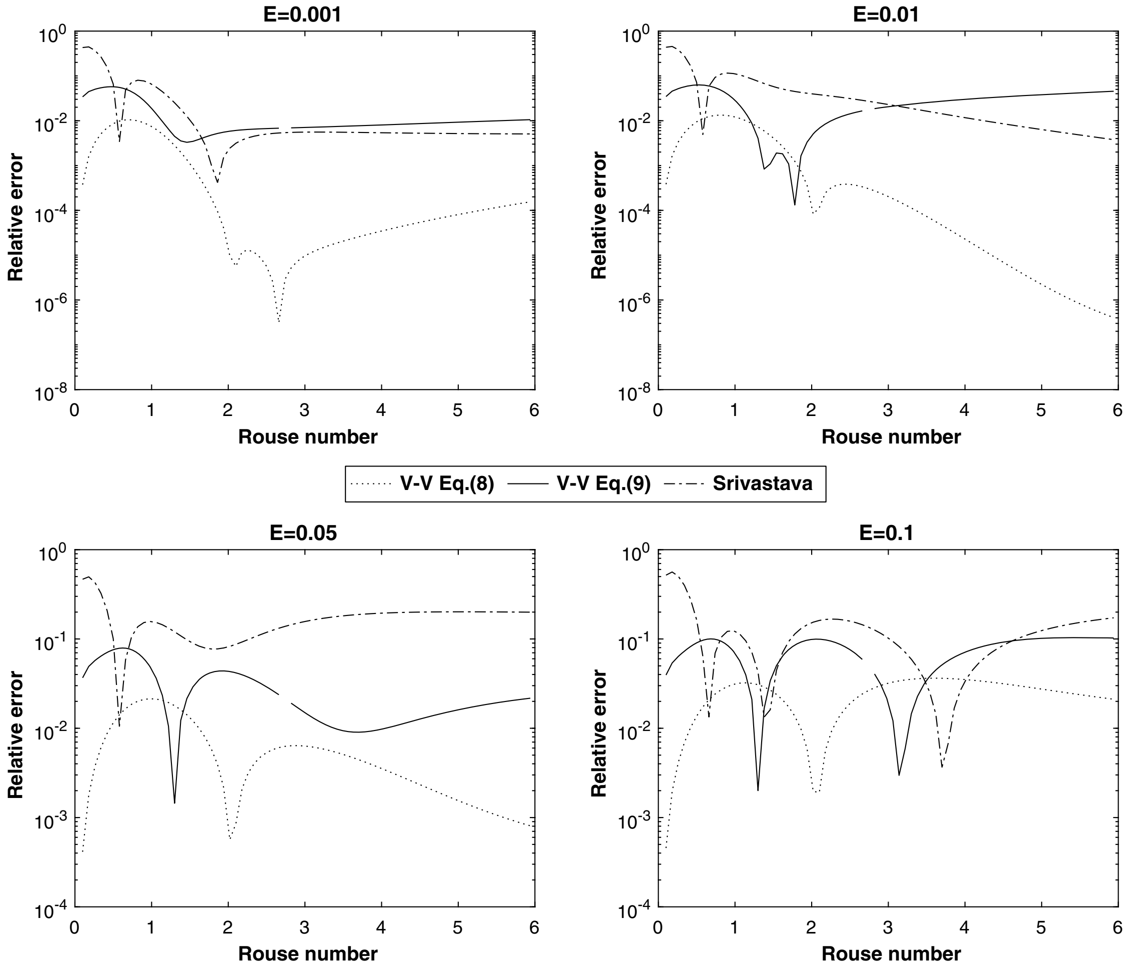

Fig. 2. Relative error (%) in the computation of J_1 using the proposed expressions by the discussers compared with those of Srivastava's method. V-V indicates the equation by the discussers.

Table 1. Comparison of J_1 values computed by methods of discussers and Guo and Julien (2004)

Method	$z = 0.55$	$z = 1.55$	$z = 2.55$	$z = 3.55$	$z = 4.55$
Exact value	9.7458×10^{-1}	2.7326	1.3002×10^{1}	7.7623×10^{1}	5.1935×10^{2}
Guo and Julien (2004)	9.7458×10^{-1}	2.7326	1.3002×10^{1}	7.7623×10^{1}	5.1935×10^{2}
Discussers' Eq. (8)	9.7443×10^{-1}	2.7319	1.3005×10^{1}	7.7651×10^{1}	5.1951×10^{2}
Discussers' Eq. (9)	9.7366×10^{-1}	2.7340	1.3011×10^{1}	7.7593×10^{1}	5.1886×10^{2}

Note: $E = 0.1$.

The writers implemented both formulas presented by the discussers and compared their results against a cubic Simpson method with 20,000 points; the result of the comparison can be seen in Fig. 2 and Table 1. It can be noticed in those figures, and through comparison with Fig. 1 in the original paper, that the new formulas by the discussers possess relative errors below 0.1% (in agreement with the comment of the discussers), but that they have similar accuracy as other alternative methods (such as Nakato's and Srivastava's methods). Although this is good news, the relative errors are still larger than those of Guo and Julien's (2004) method, which range on the 10^{-6} to 10^{-11} level (as discussed in the original paper). In addition, the discussers might want to show that their method is faster than alternative counterparts, which they have not done so far.

The discussers also suggest that their techniques would be more efficient because they do not have trigonometric or logarithmic functions since approximation of those functions with series would be time consuming. This is presently debatable because intrinsic, system-defined functions are often implemented at a low level, and they are specially handled by the compiler; consequently, they are computationally very fast (Intel 2017). In order to just give a sense of the computational speed, the writers wrote simple tests in MATLAB with the following pseudo-code:

Algorithm 1. Pseudo-code for testing computational speed of intrinsic functions versus their series-based implemented counterpart in a compiler

Generate an array $\rightarrow x = 0.1{:}0.0001{:}0.9$
    For $i = 1{:}1{,}000$
Calculate $\sin x$ and $\cos x$ with four terms Taylor series and record time $\rightarrow t_1$
    For $i = 1{:}1{,}000$
Calculate $\sin x$ and $\cos x$ with intrinsic functions and record time $\rightarrow t_2$
Report time ratio $= t_1/t_2$

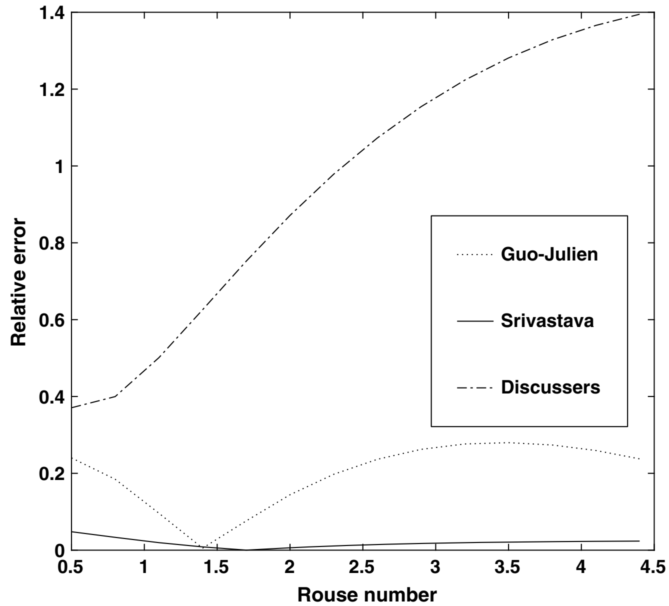

Fig. 3. Relative error (%) of various explicit closures to account for infinite summation in Guo and Julien's (2004) method.

The time ratio for MATLAB is in the order of 22, which shows that the discussers' claim should be taken with caution. Tests were conducted on the same machine and same platform.

In the final section of their discussion, the discussers analyzed the explicit closure by Guo and Julien [Eq. (10) in Zamani et al. (2017)]. They presented a regression on α_1, α_2, and α_3 in the following formula by Guo and Julien (2004):

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right) \approx \frac{\alpha_1 z}{(\alpha_2+z)^{\alpha_3}} \quad (1)$$

suggesting the following values for the coefficients: $\alpha_1 = 1.672$ [1.645 in Guo and Julien (2004)], $\alpha_2 = 1.035$ [1 in Guo and Julien (2004)], and $\alpha_3 = 0.730$ [0.716 in Guo and Julien (2004)]. The discussers claim that this formula is more accurate than both the former explicit closures by Guo and Julien (2006) and Srivastara (2006) [Eqs. (10) and (17) in Zamani et al. (2017)]. The writers respectfully have two disagreements with the discussers regarding this idea as well. First, Srivastava used a well-established perturbation technique to restructure Guo and Julien's formulae and found a new form for the summation of the infinite series; his results have tangible improvements, which can be seen in Table 2 of the original paper (Shanks 1955; Weniger 1989; Zamani 2015, p. 238). Instead, the discussers only used a regression to fit Guo and Julien's coefficients. Second, although the discussers claim their new formula is more accurate and has the maximum error of 0.3%, the test the writers set shows the opposite. Fig. 3 shows the percentage error of the closures versus the exact value of the infinite series calculated with a very large number of terms.

In conclusion, the methods presented by the discussers has a relatively small error, but they have a larger relative error than Guo and Julien's method, which does not change the conclusions of the original paper. Further, the discussers have not shown any superiority of their method over Guo and Julien's counterpart in terms of computational speed.

Data Availability Statement

The MATLAB scripts can be found in the following repository: https://github.com/kavehzamani/Einstein_sediment_integral.

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