Solutions Manual
of
Erosion and Sedimentation

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Erosion and Sedimentation

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Contents

Chapter 2

Physical Properties and Dimensional Analysis

Problem 2.1

Determine the mass density, specific weight, dynamic viscosity, and kinematic viscosity of clear water at 20°C (a) in SI units and (b) in the English system units.

Solution (a) From Table 2.3, we have, in SI units, that

$$\rho = 998 \text{ kg/m}^3, \quad \gamma = 9790 \text{ N/m}^3, \quad \mu = 10^{-3} \text{ N-s/m}^2, \quad \nu = 10^{-6} \text{ m}^2\text{/s}$$

(b) The above can be converted in the English system units as

$$\rho = 998 \, \frac{\text{kg}}{\text{m}^3} \left(\frac{1 \text{ slug}}{14.59 \text{ kg}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^3 = 1.94 \text{ slug/ft}^3$$

$$\gamma = 9790 \, \frac{\text{N}}{\text{m}^3} \left(\frac{1 \text{ lb}}{4.448 \text{ N}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^3 = 62.3 \text{ lb/ft}^3$$

$$\mu = 10^{-3} \frac{\text{N-s}}{\text{m}^2} \left(\frac{1 \text{ lb}}{4.448 \text{ N}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^2 = 2.1 \times 10^{-5} \text{ lb-s/ft}^2$$

$$\nu = 10^{-6} \frac{\text{m}^2}{\text{s}} \left(\frac{1 \text{ ft}}{0.3048 \text{ m}} \right)^2 = 1.1 \times 10^{-5} \text{ ft}^2\text{/s}$$

Problem 2.2

Determine the sediment size, mass density, specific weight, and submerged specific weight of small quarts cobbles (a) in SI units and (b) in the English system of units.

Solution (a) From Table 2.4, for small cobbles in SI units, we have

$$d_s = 64 - 128 \text{ mm}$$

From Page 9, we have

$$\rho_s = 2650 \text{ kg/m}^3, \quad \gamma_s = 26.0 \text{ kN/m}^3, \quad \gamma_s' = \gamma_s - \gamma = 26 - 9.81 = 16.19 \text{ kN/m}^3$$

(b) For the English system units, we have

$$d_s = 2.5 - 5 \text{ in}$$

$$\rho_s = 2650 \frac{\text{kg}}{\text{m}^3} \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^3 \left(\frac{1 \text{ slug}}{14.59 \text{ kg}} \right) = 5.14 \text{ slug/ft}^3$$

$$\gamma_s = 26000 \frac{\text{N}}{\text{m}^3} \left(\frac{1 \text{ lb}}{4.448 \text{ N}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^3 = 165.5 \text{ lb/ft}^3$$

$$\gamma_s' = 16.19 \frac{\text{N}}{\text{m}^3} \left(\frac{1 \text{ lb}}{4.448 \text{ N}} \right) \left(\frac{0.3048 \text{ m}}{1 \text{ ft}} \right)^3 = 103.1 \text{ lb/ft}^3$$

Problem 2.3

The volumetric sediment concentration of a sample is $C_v = 0.05$. Determine the corresponding porosity p_0; void ratio e; specific weight γ_m; specific mass ρ_m; dry specific weight γ_{md}; and dry specific mass ρ_{md}.

Solution

$$p_0 = 1 - C_v = 1 - 0.05 = 0.95$$

$$e = \frac{p_0}{C_v} = \frac{0.95}{0.05} = 19$$

$$\gamma_m = \gamma + (\gamma_s - \gamma) C_v = 9810 + (26000 - 9810)(0.05) = 10620 \text{ N/m}^3 = 10.6 \text{ kN/m}^3$$

$$\rho_m = \rho + (\rho_s - \rho) C_v = 1000 + (2650 - 1000)(0.05) = 1082.5 \text{ kg/m}^3$$

$$\gamma_{md} = \gamma_s C_v = (26000)(0.05) = 1300 \text{ N/m}^3$$

$$\rho_{md} = \rho_s C_v = (2650)(0.05) = 132.5 \text{ kg/m}^3$$

Problem 2.4

A 50-g bed-sediment sample is analyzed for particle size distribution.

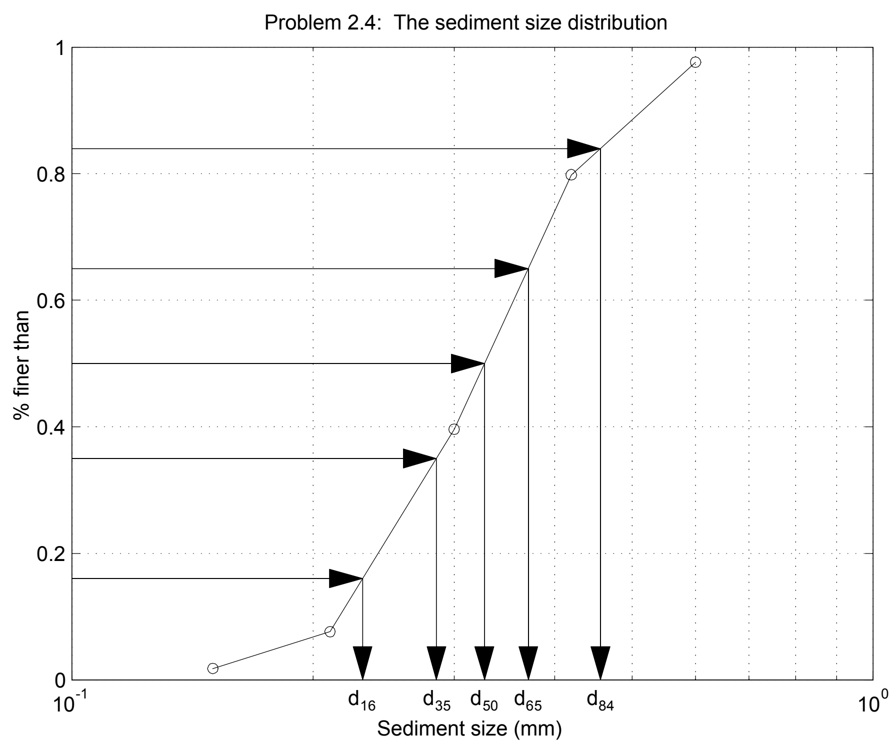
(a) Plot the sediment size distribution;

(b) determine d_{16}, d_{35}, d_{50}, d_{65}, and d_{84}; and

(c) calculate the gradation coefficient σ_g and Gr.

Solution (a) Plot the sediment size distribution.

Size fraction (mm)	Weight (mg)
$d_s \leq 0.15$	900
$0.15 < d_s \leq 0.21$	2,900
$0.21 < d_s \leq 0.30$	16,000
$0.30 < d_s \leq 0.42$	20,100
$0.42 < d_s \leq 0.60$	8,900
$0.60 \leq d_s$	1,200



(b) From the above plot, we have

$$\begin{aligned} d_{16} &= 0.231 \text{ mm}, \quad d_{35} = 0.285 \text{ mm}, \quad d_{50} = 0.327 \text{ mm} \\ d_{65} &= 0.371 \text{ mm}, \quad d_{84} = 0.457 \text{ mm} \end{aligned}$$

(c) Based on the above values, we have

$$\sigma_g = \left(\frac{d_{84}}{d_{16}}\right)^{\frac{1}{2}} = \left(\frac{0.457}{0.231}\right)^{\frac{1}{2}} = 1.41$$

$$Gr = \frac{1}{2}\left(\frac{d_{50}}{d_{16}} + \frac{d_{84}}{d_{50}}\right) = \frac{1}{2}\left(\frac{0.327}{0.231} + \frac{0.457}{0.327}\right) = 1.41$$

Problem 2.5

Consider energy losses in a straight open-channel. The energy gradient $\Delta H_L/X_c$ in a smooth channel with turbulent flow depends on the mean flow velocity V, the flow depth h, the gravitational acceleration g, the mass density ρ, and the dynamic viscosity μ. Determine the general form of the energy gradient equation from dimensional analysis.

Solution *Step 1*: According to the problem, we have

$$\frac{\Delta H_L}{X_c} = F(V, h, g, \rho, \mu) \tag{2.1}$$

in which

$\Delta H_L/X_c$	[1]
V	[L/T]
h	[L]
g	[L/T^2]
ρ	[M/L^3]
μ	[ML^{-1}T^{-1}]

Step 2: The left-hand side of (2.1) is a dimensionless variable, and we leave it alone. The right-hand side of equation (2.1) involves 5 variables which relate to three basic dimensions: T, L, and M. Therefore, we have $5-3=2$ dimensionless variables. Let us pick up V, h, and ρ as repeated variables, then we have

$$\begin{array}{lcl} V = L/T & & L = h \\ h = L & \text{which give that} & T = h/V \\ \rho = M/L^3 & & M = \rho h^3 \end{array}$$

Step 3: Express g and μ in terms of V, h, and ρ.

$$\Pi_1 : \quad g = \frac{L}{T^2} = \frac{h}{(h/V)^2} = \frac{V^2}{h}$$

or

$$\Pi_1 = \frac{V^2}{gh} \qquad \text{or} \qquad \Pi_1 = \frac{V}{\sqrt{gh}} \equiv \text{Fr} = \text{Froude number} \tag{2.2}$$

$$\Pi_2 : \quad \mu = \frac{M}{LT} = \frac{\rho h^3}{h\,(h/V)} = \rho h V$$

or

$$\Pi_2 = \frac{\rho h V}{\mu} \equiv \text{Re} = \text{Reynolds number} \tag{2.3}$$

Step 4: Using Π_1 and Π_2 to replace the variables on the right-hand side of (2.1) gives that

$$\frac{\Delta H_L}{X_c} = F(\text{Re, Fr})$$

Problem 2.6

Consider a near bed turbulent velocity profile. The time-averaged velocity v at a distance y from the bed depends on the bed-material size d_s, the flow depth h, the dynamic viscosity μ, the mass density ρ, and the boundary shear stress τ_0. Use the method of dimensional analysis to obtain a complete set of dimensionless parameters. [*Hint*: Select h, ρ, and τ_0 as repeated variables, and the problem reduces to a kinematic problem after defining the shear velocity $u_* = \sqrt{\tau_0/\rho}$ and $\nu = \mu/\rho$.]

Solution *Step 1*: According to the problem, we have

$$v = F(y, d_s, h, \mu, \rho, \tau_0) \tag{2.4}$$

Step 2: Choose h, μ, and ρ as repeated variables, then we have $7 - 3 = 4$ dimensionless variables. For the velocity v, we have

$$\begin{array}{l} h = L \\ \mu = \frac{M}{LT} \\ \rho = \frac{M}{L^3} \end{array} \quad \Longrightarrow \quad \begin{array}{l} L = h \\ T = \frac{\rho h^2}{\mu} \\ M = \rho h^3 \end{array}$$

Step 3: Dimensionless variables are

$$\begin{aligned} \Pi_1 &= \frac{vT}{L} = \frac{v\rho h^2}{\mu h} = \frac{\rho v h}{\mu} = \mathrm{Re} \\ \Pi_2 &= \frac{y}{L} = \frac{y}{h} \\ \Pi_3 &= \frac{d_s}{L} = \frac{d_s}{h} \\ \Pi_4 &= \frac{\tau_0 L T^2}{M} = \frac{\tau_0 h \rho^2 h^4}{\mu^2 \rho h^3} = \frac{\tau_0 \rho h^2}{\mu^2} \end{aligned}$$

Step 4: After defining $u_* = \sqrt{\tau_0/\rho}$ and $\nu = \mu/\rho$, the above dimensionless Π_4 can be written as

$$\Pi_4 = \frac{\tau_0 \rho h^2}{\mu^2} = \frac{\rho^2 u_*^2 h^2}{(\rho\nu)^2} = \left(\frac{u_* h}{\nu}\right)^2$$

For simplicity, we can write Π_4 as

$$\Pi_4 = \frac{u_* h}{\nu}$$

Step 5: Finally, we have

$$\mathcal{F}(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0$$

or

$$\mathcal{F}\left(\mathrm{Re}, \frac{y}{h}, \frac{d_s}{h}, \frac{u_* h}{\nu}\right) = 0$$

Note: From Π_3 and Π_4, we can get a new dimensionless variable $\frac{u_* d_s}{\nu}$. We can use this dimensionless to replace Π_4.

Chapter 3

Mechanics of Sediment-Laden Flows

Exercises

1. Demonstrate, using equations (3.6) and (3.4), that div $\boldsymbol{\omega} = 0$ for homogeneous incompressible fluids.

   Solution

$$\otimes_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \quad \otimes_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \quad \otimes_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

$$\begin{aligned}
\text{div } \boldsymbol{\omega} &= \frac{\partial \otimes_x}{\partial x} + \frac{\partial \otimes_y}{\partial y} + \frac{\partial \otimes_z}{\partial z} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\
&= \cancel{\frac{\partial^2 v_z}{\partial x \partial y}} - \cancel{\cancel{\frac{\partial^2 v_y}{\partial x \partial z}}} + \frac{\partial^2 v_x}{\partial y \partial z} - \cancel{\frac{\partial^2 v_z}{\partial x \partial y}} + \cancel{\cancel{\frac{\partial^2 v_y}{\partial x \partial z}}} - \frac{\partial^2 v_x}{\partial y \partial z} \\
&= 0
\end{aligned}$$

2. Demonstrate that equations (3.8a) and (3.9a) reduce to equation (3.11a).

Solution From equation (3.8a), we have

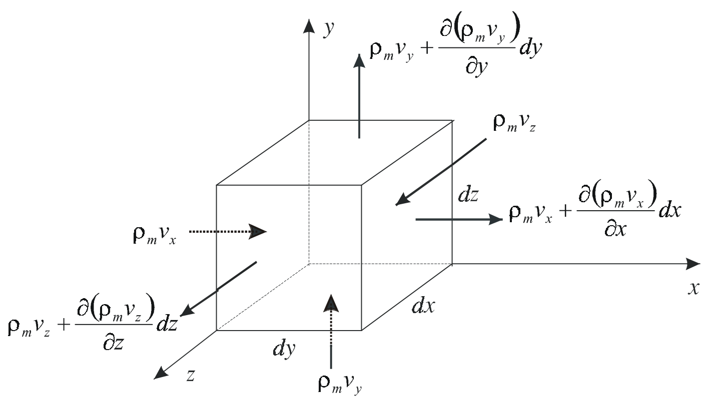
$$\begin{aligned}
a_x &= \frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + \ominus_x \frac{dx}{dt} + \frac{\oslash_z}{2}\frac{dy}{dt} + \frac{\oslash_y}{2}\frac{dz}{dt} + \frac{1}{2}\left(\otimes_y \frac{dz}{dt} - \otimes_z \frac{dy}{dt}\right) \\
&= \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \frac{v_y}{2}\left(\cancel{\frac{\partial v_y}{\partial x}} + \frac{\partial v_x}{\partial y}\right) + \frac{v_z}{2}\left(\cancel{\cancel{\frac{\partial v_z}{\partial x}}} + \frac{\partial v_x}{\partial z}\right) \\
&\quad + \frac{1}{2}\left[v_z\left(\frac{\partial v_x}{\partial z} - \cancel{\cancel{\frac{\partial v_z}{\partial x}}}\right) - v_y\left(\cancel{\frac{\partial v_y}{\partial x}} - \frac{\partial v_x}{\partial y}\right)\right] \\
&= \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}
\end{aligned}$$

From (3.9a), we have

$$\begin{aligned}
a_x &= \frac{\partial v_x}{\partial t} + v_z \otimes_y - v_y \otimes_z + \frac{\partial (v^2/2)}{\partial x} \\
&= \frac{\partial v_x}{\partial t} + v_z\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - v_y\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{v_x^2 + v_y^2 + v_z^2}{2}\right) \\
&= \frac{\partial v_x}{\partial t} + v_z \frac{\partial v_x}{\partial z} - v_z \cancel{\frac{\partial v_z}{\partial x}} - v_y \cancel{\cancel{\frac{\partial v_y}{\partial x}}} + v_y \frac{\partial v_x}{\partial y} + v_x \frac{\partial v_x}{\partial x} + v_y \cancel{\cancel{\frac{\partial v_y}{\partial x}}} + v_z \cancel{\frac{\partial v_z}{\partial x}} \\
&= \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}
\end{aligned}$$

3. Demonstrate that continuity relationship [Eq. (3.10a)] in Cartesian coordinates by considering the integral mass change and the balance of mass fluxes entering and leaving a cubic control volume element.

Solution Consider the flow into and out of an element volume.



The net rate of flow into the element in the x direction is

$$-\frac{\partial(\rho_m v_x)}{\partial x} dxdydz$$

The net rate of flow into the element in the y direction is

$$-\frac{\partial(\rho_m v_y)}{\partial y} dxdydz$$

The net rate of flow into the element in the z direction is

$$-\frac{\partial(\rho_m v_z)}{\partial z} dxdydz$$

The net rate of flow into the element is

$$-\left[\frac{\partial(\rho_m v_x)}{\partial x} + \frac{\partial(\rho_m v_y)}{\partial y} + \frac{\partial(\rho_m v_z)}{\partial z}\right] dxdydz$$

The rate of increase of mass inside the element is

$$\frac{\partial}{\partial t}(\rho_m dxdydz) = \frac{\partial \rho_m}{\partial t} dxdydz$$

The law of mass conservation states that the mass rate into the element should equal the rate of increase of mass inside the element, i.e.

$$-\left[\frac{\partial(\rho_m v_x)}{\partial x} + \frac{\partial(\rho_m v_y)}{\partial y} + \frac{\partial(\rho_m v_z)}{\partial z}\right] dxdydz = \frac{\partial \rho_m}{\partial t} dxdydz$$

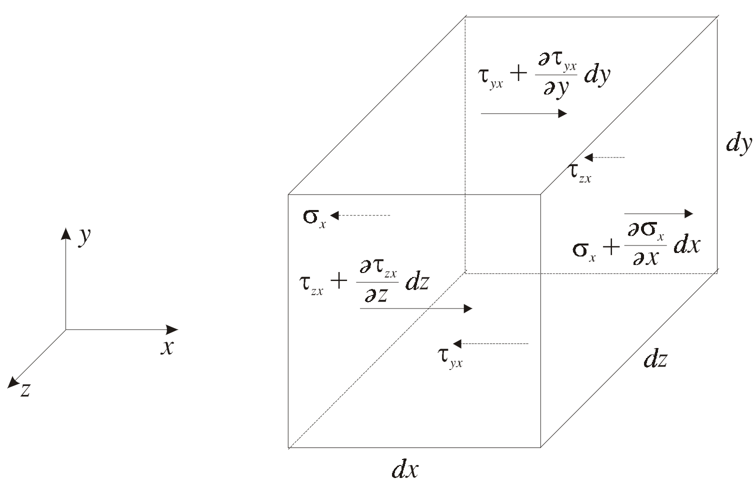
or

$$\boxed{\frac{\partial \rho_m}{\partial t} + \frac{\partial(\rho_m v_x)}{\partial x} + \frac{\partial(\rho_m v_y)}{\partial y} + \frac{\partial(\rho_m v_z)}{\partial z} = 0}$$

4. Derive the x component of the equation of motion in Cartesian coordinates [Eq. (3.14a)] from the force diagram in Figure 3.4.

Solution The gravity components in the x direction is

$$\rho_m g_x dxdydz \tag{3.1}$$



The net surface forces in the x direction are

$$\frac{\partial \sigma_x}{\partial x} dxdydz + \frac{\partial \tau_{yx}}{\partial y} dydxdz + \frac{\partial \tau_{zx}}{\partial z} dzdxdy \tag{3.2}$$

The mass of the fluid element is

$$\rho_m dxdydz \tag{3.3}$$

Applying (3.1), (3.2) and (3.3) to Newton's second law: $\mathbf{F} = m\mathbf{a}$, in the x direction, we have

$$\rho_m g_x dxdydz + \frac{\partial \sigma_x}{\partial x} dxdydz + \frac{\partial \tau_{yx}}{\partial y} dydxdz + \frac{\partial \tau_{zx}}{\partial z} dzdxdy = \rho_m dxdydz a_x$$

Finally, we get

$$\boxed{a_x = g_x + \frac{1}{\rho_m}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)}$$

5. Derive the x component of the Bernoulli equation [Eq. (3.19a)] from equations (3.9a), (3.14a), and (3.18a).

   Solution Assumptions: (1) incompressible (not necessary), (2) steady (not necessary), (3) irrotational, and (4) frictionless. Euler equation in the x direction for incompressible fluids is

$$a_x = g_x - \frac{1}{\rho_m}\frac{\partial p}{\partial x} = g_x - \frac{\partial}{\partial x}\left(\frac{p}{\rho_m}\right) \tag{3.4}$$

in which

$$\begin{aligned} a_x &= \cancel{\frac{\partial v_x}{\partial t}} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ &= v_x \frac{\partial v_x}{\partial x} + \underline{v_y \frac{\partial v_y}{\partial x}} + v_y\left(\frac{\partial v_x}{\partial y} - \underline{\frac{\partial v_y}{\partial x}}\right) + \underline{\underline{v_z \frac{\partial v_z}{\partial x}}} + v_z\left(\frac{\partial v_x}{\partial z} - \underline{\underline{\frac{\partial v_z}{\partial x}}}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{v_x^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{v_y^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{v_z^2}{2}\right) - v_y \cancel{\otimes_z} + v_z \cancel{\otimes_y} \\ &= \frac{\partial}{\partial x}\left(\frac{v_x^2 + v_y^2 + v_z^2}{2}\right) = \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) \qquad \underline{\text{Note: } v^2 = v_x^2 + v_y^2 + v_z^2} \end{aligned} \tag{3.5}$$

Since gravity is conservative, we can define a gravitation potential as

$$\Omega_g = -g\hat{z} \tag{3.6}$$

in which $\hat{z}$ is vertical upward. Thus,

$$g_x = \frac{\partial \Omega_g}{\partial x} \tag{3.7}$$

Substituting (3.5) and (3.7) into (3.4) gives that

$$\frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) = \frac{\partial \Omega_g}{\partial x} - \frac{\partial}{\partial x}\left(\frac{p}{\rho_m}\right)$$

or

$$\frac{\partial}{\partial x}\left(-\Omega_g + \frac{p}{\rho_m} + \frac{v^2}{2}\right) = 0 \tag{3.8}$$

Similarly, we can write Euler equations in the y and z directions as

$$\frac{\partial}{\partial y}\left(-\Omega_g + \frac{p}{\rho_m} + \frac{v^2}{2}\right) = 0 \tag{3.9}$$

$$\frac{\partial}{\partial z}\left(-\Omega_g + \frac{p}{\rho_m} + \frac{v^2}{2}\right) = 0 \tag{3.10}$$

Finally, from (3.8) to (3.10), we get

$$-\Omega_g + \frac{p}{\rho_m} + \frac{v^2}{2} = \text{const}$$

Substituting (3.6) into the above equation gives that

$$g\hat{z} + \frac{p}{\rho_m} + \frac{v^2}{2} = \text{const}$$

or

$$\boxed{\hat{z} + \frac{p}{\gamma_m} + \frac{v^2}{2g} = \text{const}} \tag{3.11}$$

in which the left-hand side is called Bernoulli sum.

6. Derive the x component of the momentum equation [Eq. (3.27a)] from Equation (3.25).

Solution

$$\begin{aligned} &\int_\forall \rho_m \left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) d\forall \\ = \;&\int_\forall \rho_m g_x d\forall - \int_\forall \frac{\partial p}{\partial x} d\forall + \int_\forall \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) d\forall \end{aligned}$$

By virtue of the continuity equation, the integrand of the left-hand side can be written as

$$
\begin{aligned}
& \rho_m \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \\
= & \frac{\partial (\rho_m v_x)}{\partial t} + \frac{\partial \rho_m (v_x v_x)}{\partial x} + \frac{\partial (\rho_m v_x v_y)}{\partial y} + \frac{\partial (\rho_m v_x v_z)}{\partial z} \\
& -v_x \left(\cancel{\frac{\partial \rho_m}{\partial t}} + \cancel{\frac{\partial (\rho_m v_x)}{\partial x}} + \cancel{\frac{\partial (\rho_m v_y)}{\partial y}} + \cancel{\frac{\partial (\rho_m v_z)}{\partial z}} \right) \\
= & \frac{\partial (\rho_m v_x)}{\partial t} + \frac{\partial \rho_m (v_x v_x)}{\partial x} + \frac{\partial (\rho_m v_x v_y)}{\partial y} + \frac{\partial (\rho_m v_x v_z)}{\partial z}
\end{aligned}
$$

then applying the divergence theorem, we get

$$
\begin{aligned}
& \int_\forall \rho_m \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) d\forall \\
= & \int_\forall \left(\frac{\partial (\rho_m v_x)}{\partial t} + \frac{\partial \rho_m (v_x v_x)}{\partial x} + \frac{\partial (\rho_m v_x v_y)}{\partial y} + \frac{\partial (\rho_m v_x v_z)}{\partial z} \right) d\forall \\
= & \frac{\partial}{\partial t} \int_\forall \rho_m v_x d\forall + \int_A \rho_m v_x \left(v_x \frac{\partial x}{\partial n} + v_y \frac{\partial y}{\partial n} + v_z \frac{\partial z}{\partial n} \right) dA
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \int_\forall \rho_m g_x d\forall - \int_\forall \frac{\partial p}{\partial x} d\forall + \int_\forall \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) d\forall \\
= & \int_\forall \rho_m g_x d\forall - \int_A p \frac{\partial x}{\partial n} dA + \int_A \left(\tau_{xx} \frac{\partial x}{\partial n} + \tau_{yx} \frac{\partial y}{\partial n} + \tau_{zx} \frac{\partial z}{\partial n} \right) dA
\end{aligned}
$$

From the above two equations, we have

$$
\begin{aligned}
& \frac{\partial}{\partial t} \int_\forall \rho_m v_x d\forall + \int_A \rho_m v_x \left(v_x \frac{\partial x}{\partial n} + v_y \frac{\partial y}{\partial n} + v_z \frac{\partial z}{\partial n} \right) dA \\
= & \int_\forall \rho_m g_x d\forall - \int_A p \frac{\partial x}{\partial n} dA + \int_A \left(\tau_{xx} \frac{\partial x}{\partial n} + \tau_{yx} \frac{\partial y}{\partial n} + \tau_{zx} \frac{\partial z}{\partial n} \right) dA
\end{aligned}
$$

7. Demonstrate that the power equation [Eq. (3.30)] stems from Equation (3.28).

Solution (1) Consider the left-hand side of the equation.

$$
\begin{aligned}
a_x v_x &= v_x \frac{\partial v_x}{\partial t} + v_x \left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \\
&= \frac{\partial}{\partial t} \left(\frac{v_x^2}{2} \right) + v_x \frac{\partial}{\partial x} \left(\frac{v_x^2}{2} \right) + v_y \frac{\partial}{\partial y} \left(\frac{v_x^2}{2} \right) + v_z \frac{\partial}{\partial z} \left(\frac{v_x^2}{2} \right)
\end{aligned}
$$

Similarly, we have

$$a_y v_y = \frac{\partial}{\partial t}\left(\frac{v_y^2}{2}\right) + v_x \frac{\partial}{\partial x}\left(\frac{v_y^2}{2}\right) + v_y \frac{\partial}{\partial y}\left(\frac{v_y^2}{2}\right) + v_z \frac{\partial}{\partial z}\left(\frac{v_y^2}{2}\right)$$

$$a_z v_z = \frac{\partial}{\partial t}\left(\frac{v_z^2}{2}\right) + v_x \frac{\partial}{\partial x}\left(\frac{v_z^2}{2}\right) + v_y \frac{\partial}{\partial y}\left(\frac{v_z^2}{2}\right) + v_z \frac{\partial}{\partial z}\left(\frac{v_z^2}{2}\right)$$

Thus,

$$a_x v_x + a_y v_y + a_z v_z = \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + v_x \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) + v_y \frac{\partial}{\partial y}\left(\frac{v^2}{2}\right) + v_z \frac{\partial}{\partial z}\left(\frac{v^2}{2}\right)$$

in which $v^2 = v_x^2 + v_y^2 + v_z^2$. Then

$$\begin{aligned}
&\int_\forall \rho_m (a_x v_x + a_y v_y + a_z v_z) d\forall \\
=\ &\int_\forall \rho_m \frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) d\forall + \int_\forall \rho_m \left[v_x \frac{\partial}{\partial x}\left(\frac{v^2}{2}\right) + v_y \frac{\partial}{\partial y}\left(\frac{v^2}{2}\right) + v_z \frac{\partial}{\partial z}\left(\frac{v^2}{2}\right)\right] d\forall \\
=\ &\frac{\partial}{\partial t}\int_\forall \rho_m \frac{v^2}{2} d\forall + \int_A \rho_m \left(\frac{v^2}{2}\right)\left(v_x \frac{\partial x}{\partial n} + v_y \frac{\partial y}{\partial n} + v_z \frac{\partial z}{\partial n}\right) dA \qquad (3.12)
\end{aligned}$$

(2) Consider the body force term.

$$\begin{aligned}
&\int_\forall \rho_m (v_x g_x + v_y g_y + v_z g_z) d\forall = \int_\forall \rho_m (v_x \frac{\partial \Omega_g}{\partial x} + v_y \frac{\partial \Omega_g}{\partial y} + v_z \frac{\partial \Omega_g}{\partial z}) d\forall \\
=\ &\int_A \rho_m \Omega_g \left(v_x \frac{\partial x}{\partial n} + v_y \frac{\partial y}{\partial n} + v_z \frac{\partial z}{\partial n}\right) dA \qquad (3.13)
\end{aligned}$$

(3) Consider the surface shear term.

$$\begin{aligned}
&\int_\forall v_x \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) d\forall \\
=\ &\int_\forall \left(\frac{\partial (v_x \sigma_x)}{\partial x} + \frac{\partial (v_x \tau_{yx})}{\partial y} + \frac{\partial (v_x \tau_{zx})}{\partial z}\right) d\forall \\
&- \int_\forall \left(\sigma_x \frac{\partial v_x}{\partial x} + \tau_{yx} \frac{\partial v_x}{\partial y} + \tau_{zx} \frac{\partial v_x}{\partial z}\right) d\forall \\
=\ &\int_A \left(v_x \sigma_x \frac{\partial x}{\partial n} + v_x \tau_{yx} \frac{\partial y}{\partial n} + v_x \tau_{zx} \frac{\partial z}{\partial n}\right) dA \\
&- \int_\forall \left(\sigma_x \frac{\partial v_x}{\partial x} + \tau_{yx} \frac{\partial v_x}{\partial y} + \tau_{zx} \frac{\partial v_x}{\partial z}\right) d\forall \\
=\ &\int_A \left[v_x \sigma_x \frac{\partial x}{\partial n} + v_x \tau_{yx} \frac{\partial y}{\partial n} + v_x \tau_{zx} \frac{\partial z}{\partial n}\right] dA + \int_\forall p \frac{\partial v_x}{\partial x} d\forall \\
&- \int_\forall \left[\tau_{xx} \frac{\partial v_x}{\partial x} + \tau_{yx} \frac{\partial v_x}{\partial y} + \tau_{zx} \frac{\partial v_x}{\partial z}\right] d\forall \qquad (3.14)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_\forall v_y\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)d\forall \\
=&\int_A\left(v_y\tau_{xy}\frac{\partial x}{\partial n}+v_y\sigma_y\frac{\partial y}{\partial n}+v_y\tau_{zy}\frac{\partial z}{\partial n}\right)dA \\
&+\int p\frac{\partial v_y}{\partial y}d\forall-\int_\forall\left(\tau_{xy}\frac{\partial v_y}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}+\tau_{zy}\frac{\partial v_y}{\partial z}\right)d\forall
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
&\int_\forall v_z\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}\right)d\forall \\
=&\int_A\left(v_z\tau_{xz}\frac{\partial x}{\partial n}+v_z\tau_{yz}\frac{\partial y}{\partial n}+v_z\sigma_z\frac{\partial z}{\partial n}\right)dA \\
&+\int p\frac{\partial v_z}{\partial z}d\forall-\int_\forall\left(\tau_{xz}\frac{\partial v_z}{\partial x}+\tau_{yz}\frac{\partial v_z}{\partial y}+\tau_{zz}\frac{\partial v_z}{\partial z}\right)d\forall
\end{aligned}\tag{3.16}$$

From(3.14), (3.15) and (3.16), we have

$$\begin{aligned}
&\int_\forall\left[v_x\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+v_y\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)\right. \\
&\left.+v_z\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}\right)\right]d\forall \\
=&\int_A\left[\left(v_x\sigma_x+v_y\tau_{xy}+v_z\tau_{xz}\right)\frac{\partial x}{\partial n}+\left(v_x\tau_{yx}+v_y\sigma_y+v_z\tau_{yz}\right)\frac{\partial y}{\partial n}\right. \\
&\left.+\left(v_x\tau_{zx}+v_y\tau_{zy}+v_z\sigma_z\right)\frac{\partial z}{\partial n}\right]dA \\
&+\int_\forall p\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)d\forall-\int_\forall\left[\tau_{xx}\frac{\partial v_x}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}+\tau_{zz}\frac{\partial v_z}{\partial z}\right. \\
&\left.+\tau_{xy}\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)+\tau_{xz}\left(\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z}\right)+\tau_{yz}\left(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}\right)\right]d\forall
\end{aligned}$$

Define

$$\rho_m\frac{d\Omega_e}{dt}=-p\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

then

$$-\int_\forall p\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)d\forall=\frac{\partial}{\partial t}\int_\forall\rho_m\Omega_e d\forall+\int_A\rho_m\Omega_e\left(v_x\frac{\partial x}{\partial n}+v_y\frac{\partial y}{\partial n}+v_z\frac{\partial z}{\partial n}\right)dA$$

Note that the left side hand is zero for incompressible fluids. Since

$$\oslash_z=\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x},\quad \oslash_y=\frac{\partial v_z}{\partial x}+\frac{\partial v_x}{\partial z},\quad \oslash_x=\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}$$

then

$$\int_{\forall}\left[v_x\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right)+v_y\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}\right)\right.$$
$$\left.+v_z\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right)\right]d\forall$$
$$=\int_A\left[\left(v_x\sigma_x+v_y\tau_{xy}+v_z\tau_{xz}\right)\frac{\partial x}{\partial n}+\left(v_x\tau_{yx}+v_y\sigma_y+v_z\tau_{yz}\right)\frac{\partial y}{\partial n}\right.$$
$$\left.+\left(v_x\tau_{zx}+v_y\tau_{zy}+v_z\sigma_z\right)\frac{\partial z}{\partial n}\right]dA-\frac{\partial}{\partial t}\int_{\forall}\rho_m\Omega_e d\forall$$
$$-\int_A\rho_m\Omega_e\left(v_x\frac{\partial x}{\partial n}+v_y\frac{\partial y}{\partial n}+v_z\frac{\partial z}{\partial n}\right)dA-\int_{\forall}\left(\tau_{xx}\frac{\partial v_x}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}\right.$$
$$\left.+\tau_{zz}\frac{\partial v_z}{\partial z}+\tau_{xy}\oslash_z+\tau_{xz}\oslash_y+\tau_{yz}\oslash_x\right)d\forall \tag{3.17}$$

(4) Combining (3.12), (3.13) and (3.17) and rearranging them, we have

$$\frac{\partial}{\partial t}\int_{\forall}\rho_m\left(\frac{v^2}{2}-\Omega_g+\Omega_e\right)d\forall$$
$$+\int_A\rho_m\left(\frac{v^2}{2}-\Omega_g+\Omega_e\right)\left(v_x\frac{\partial x}{\partial n}+v_y\frac{\partial y}{\partial n}+v_z\frac{\partial z}{\partial n}\right)dA$$
$$=+\int_A\left[\left(v_x\sigma_x+v_y\tau_{xy}+v_z\tau_{xz}\right)\frac{\partial x}{\partial n}+\left(v_x\tau_{yx}+v_y\sigma_y+v_z\tau_{yz}\right)\frac{\partial y}{\partial n}\right.$$
$$\left.+\left(v_x\tau_{zx}+v_y\tau_{zy}+v_z\sigma_z\right)\frac{\partial z}{\partial n}\right]d\forall$$
$$-\int_{\forall}\left(\tau_{xx}\frac{\partial v_x}{\partial x}+\tau_{yy}\frac{\partial v_y}{\partial y}+\tau_{zz}\frac{\partial v_z}{\partial z}+\tau_{xy}\oslash_z+\tau_{xz}\oslash_y+\tau_{yz}\oslash_x\right)d\forall$$

Note that since $\Omega_g \neq f(t)$, we have $\frac{\partial}{\partial t}\int_{\forall}\rho_m\Omega_g d\forall = 0$.

8. Demonstrate, from the specific energy E, that $q^2 = gh_c^3$ and $E_{\min} = 3h_c/2$ for steady one-dimensional open-channel flow.

Solution

$$E = h + \frac{V^2}{2g} = h + \frac{q^2}{2gh^2} \tag{3.18}$$

When $E = E_{\min}$, we have

$$\frac{dE}{dh} = 1 - \frac{q^2}{gh^3} = 0$$

which gives that

$$q^2 = gh_c^3$$

Substituting the above equation into (3.18) gives that

$$E = h_c + \frac{1}{2}h_c = \frac{3}{2}h_c$$

Problem 3.1

With reference to Figure 3.2, determine which type of deformation is obtained when $v_x = 2y$ and $v_y = v_z = 0$. With x to the right, y up, z must come out of the $x - y$ plane.

Solution (1) Translation along x only. (2) Check linear deformation:

$$\ominus_x = \frac{\partial v_x}{\partial x} = 0, \quad \ominus_y = \ominus_z = 0$$

So there are no linear deformations. (3) Check angular deformation:

$$\oslash_x = 0, \quad \oslash_y = 0, \quad \oslash_z = \frac{\partial v_x}{\partial y} + \cancel{\frac{\partial v_y}{\partial x}} = 2$$

So we have angular deformation $\oslash = 2$. (4) Check rotation:

$$\otimes_x = \otimes_y = 0, \quad \otimes_z = \cancel{\frac{\partial v_y}{\partial x}} - \frac{\partial v_x}{\partial y} = -2$$

So we have a clockwise rotation around the z axis.

Problem 3.2

Given the 280-ft-wide cross section (depth measurements every 10 ft) and depth-averaged velocity profile below, calculate:

(a) the total cross-sectional area, $A = \sum_i a_i$, where a_i is the incremental cross-sectional area;

(b) the total discharge, $Q = \sum_i a_i v_i$, where v_i is the depth-averaged flow velocity normal to the incremental area; and

(c) the cross-sectional average flow velocity $V = Q/A$.

Solution (a) The cross-sectional area:

$$\begin{aligned} A &= \sum_i a_i = \sum_{i=1}^{n-1} \Delta w \frac{h_i + h_{i+1}}{2} = \frac{\Delta w}{2} \sum_{i=1}^{n-1} h_i + \frac{\Delta w}{2} \sum_{i=2}^{n} h_i \\ &= \frac{\Delta w}{2}(h_1 + h_n) + \Delta w \sum_{i=2}^{n-1} h_i \end{aligned}$$

in which $\Delta w = 10$ ft $= 3.048$ m, $n = 29$, and h_i are listed in the program appended. The calculated cross-sectional area is

$$\boxed{A = 59.63 \text{ m}^2}$$

(b) The total discharge:

$$Q = \sum_{i=1}^{n-1} a_i v_i = \frac{\Delta w}{2}(h_1 v_1 + h_{n-1} v_{n-1}) + \Delta w \sum_{i=2}^{n-2} h_i v_i$$

in which v_i are measured velocity listed in the program appended. the calculated discharge is

$$\boxed{Q = 5.74 \text{ m}^3/\text{s}}$$

(c) The cross-sectional average velocity is

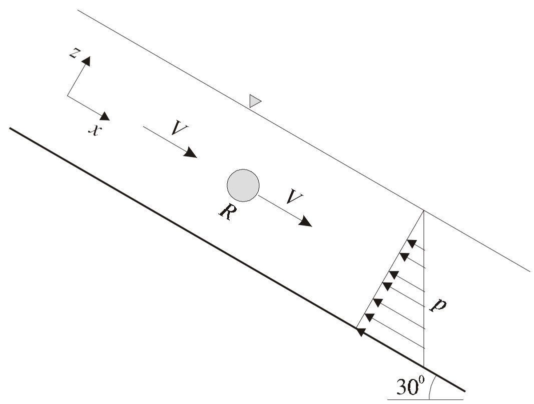
$$\boxed{V = \frac{Q}{A} = 0.096 \text{ m/s}}$$

Appendix: Measured data and MatLab program

```
%Problem 3.2
% Read the data of flow depth and velocity from the figure
h = [   0    1.49    1.72    1.90    2.35   1.99    1.90 ...
     1.89    1.92    2.33    2.47    2.40    2.01    2.19 ...
     2.91    3.58    3.81    3.77    3.77    3.19    2.90 ...
     2.90    2.54    1.89    1.80    1.57    1.37    0.59 0]; %in ft
v = [0    3.16    5.26    7.11    8.98    9.91   11.58 ...
   2.32  13.68   13.39   13.80   13.37   13.06   12.30 ...
  11.74  10.99   10.69   10.02    9.89    9.70    9.20 ...
   8.89   8.35    8.26    7.73    6.99    6.49    4.77 0]; % ft/s

% Convert English units into SI units
h1 = h.*0.3048;   % in meters
v1 = v.*0.01   % in m/s

% Calculate the cross section area
A = 10.*0.3048.*(h1(2)+h1(28))./2;
for i=2:1:28, A = A+h1(i).*3.048; end

% Calculate the total discharge
Qi = h1.*v1.*3.048; Q = 0.5.*(Qi(1)+Qi(29));
for i=2:1:28; Q = Q+Qi(i); end

V = Q./A; % Calculate the average velocity
A, Q, V, % Print the results
```

Problem 3.3

Calculate the magnitude and direction of the buoyancy force applied on a sphere submerged under steady one-dimensional flow ($v_y = v_z = 0$) on a steep slope. Assume that the particle is stationary with respect to the surrounding inviscid fluid of density ρ_m. Compare the results with Example 3.3. [*Hint*: Integrate the pressure distribution around the sphere from Equation (3.17c) with a_z.]



Solution According to Euler's equations, we have

$$0 = g \sin 30° - \frac{1}{\rho_m}\frac{\partial p}{\partial x}$$

$$0 = -g \cos 30° - \frac{1}{\rho_m}\frac{\partial p}{\partial z}$$

Thus, we have

$$dp = \rho_m \frac{\partial p}{\partial x} dx + \rho_m \frac{\partial p}{\partial z} dz = \rho_m g \sin 30° dx - \rho_m g \cos 30° dz$$

or

$$p = \rho_m g \sin 30° x - \rho_m g \cos 30° z + \text{const}$$

Since

$$x = R \sin\theta \cos\phi, \quad z = R\cos\theta$$

then

$$p = \rho_m g \sin 30° R \sin\theta \cos\phi - \rho_m g \cos 30° R \cos\theta + \text{const}$$

Since the pressure is symmetric about the z axis, the buoyance force points the z

direction. Thus,

$$\begin{aligned}
F_B &= -\int_0^{2\pi}\int_0^{\pi}\left(\rho_m g\sin 30^\circ R\sin\theta\cos\phi - \rho_m g\cos 30^\circ R\cos\theta\right)R^2\sin\theta\cos\theta d\theta d\phi \\
&= -\rho_m g R^3\sin 30^\circ\int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\cos\phi d\phi + \rho_m g R^3\cos 30^\circ\int_0^{\pi}\sin\theta\cos^2\theta d\theta\int_0^{2\pi}d\phi \\
&= 0 + \rho_m g R^3\cos 30^\circ\left(-\frac{1}{3}\cos^3\theta\right)_0^{\pi}(2\pi) \\
&= \rho_m g\left(\frac{4\pi}{3}R^3\right)\cos 30^0 = \gamma_m V_{\text{sphere}}\cos 30^\circ
\end{aligned}$$

Another method: The pressure distribution is shown in the above figure. Intuitively, the pressure distribution around the sphere is symmetrical about the z axis. Then we only have buoyancy force in the z direction. Since the pressure is larger at the bottom than that at the top, the buoyancy force must be in the positive direction of z. Analogous to Example 3.3, we have

$$F_B = -\rho_m g_z V_{\text{sphere}}$$

now

$$g_z = -g\cos 30^\circ$$

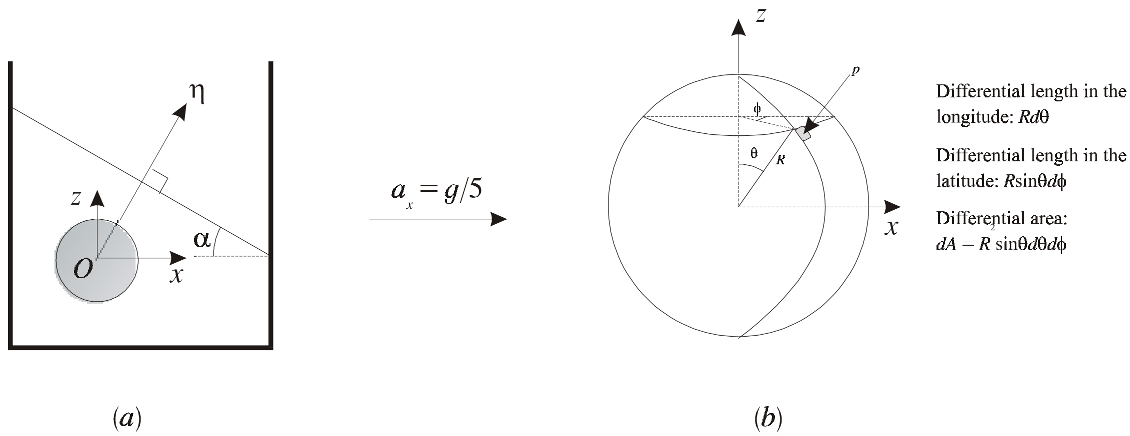
then

$$\boxed{F_B = \rho_m g V_{\text{sphere}}\cos 30^\circ}$$

Problem 3.4

A rectangular container 10 m long, 6 m wide and 4 m high in half filled with clear water. Integrate the pressure distribution to calculate the buoyancy force in newtons on a submerged sphere, 10 cm in diameter, located 1 m below the center of the container. Compare the buoyancy force under hydrostatic conditions with the case when the container is accelerated horizontally at $g/5$.

Solution Refer to the following figure (a).



Choose a Cartesian coordinates (x, z) with the container, according to Euler equations we then have

$$\begin{aligned} 0 &= -\frac{g}{5} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ 0 &= 0 - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ 0 &= -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}$$

in which the acceleration of the container is considered as an inertia force per unit mass. From the above, we have

$$\begin{aligned} dp &= \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \\ &= -\frac{1}{5}\rho g dx - \rho g dz \end{aligned}$$

or

$$p = -\frac{1}{5}\rho g x - \rho g z + \text{const} \tag{3.19}$$

Since the pressure is a constant on the free surface, we have the angle α between the free surface and the x axis is

$$\tan\alpha = \frac{1}{5} \quad \text{or} \quad \alpha = 11.3^\circ \tag{3.20}$$

Since

$$x = R\sin\theta\cos\phi, \quad z = R\cos\theta$$

Eq. (3.19) can be written as

$$p = -\frac{1}{5}\rho g R\sin\theta\cos\phi - \rho g R\cos\theta \tag{3.21}$$

Then we get

$$\begin{aligned} F_x &= -\int_A p dA \sin\theta \\ &= -\int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{5}\rho g R\sin\theta\cos\phi - \rho g R\cos\theta\right)\sin\theta R^2\sin\theta d\theta d\phi \\ &= \frac{1}{5}\rho g R^3\int_0^{2\pi}\cos^2\phi d\phi\int_0^{\pi}\sin^3\theta d\theta + \rho g R^3\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\theta\cos\theta d\theta \\ &= \frac{1}{5}\rho g R^3\left(4\cdot\frac{\pi}{2}\cdot\frac{1}{2}\right)\left(2\cdot\frac{2}{3}\right) + \rho g R^3(0) \\ &= \frac{\rho g}{5}\left(\frac{4}{3}\pi R^3\right) \end{aligned}$$

$$
\begin{aligned}
F_z &= -\int_A p dA \cos\theta \\
&= -\int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{5}\rho g R\sin\theta\cos\phi - \rho g R\cos\theta\right)\cos\theta R^2\sin\theta d\theta d\phi \\
&= \frac{1}{5}\rho g R^3\int_0^{2\pi}\cos^2\phi d\phi\int_0^{\pi}\sin^2\theta\cos\theta d\theta + \rho g R^3\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\cos^2\theta d\theta \\
&= 0 + 2\pi\rho g R^3\left(-\frac{1}{3}\cos^3\theta\right)_0^{\pi} = \rho g\left(\frac{4}{3}\pi R^3\right)
\end{aligned}
$$

Since

$$\frac{F_z}{F_x} = \frac{1}{5} = \tan\alpha$$

then we have

$$\alpha = 11.3^\circ$$

This shows that the buoyant force is perpendicular to the water surface. Then

$$
\begin{aligned}
F_B &= F_x\sin\alpha + F_z\cos\alpha \\
&= \rho g\left(\frac{4}{3}\pi R^3\right)\left(\frac{1}{5}\sin\alpha + \cos\alpha\right)
\end{aligned}
$$

or

$$\boxed{F_B = \rho g\left(\frac{4}{3}\pi R^3\right)\left(\frac{1}{5}\sin\alpha + \cos\alpha\right)}$$

Remarks: We can directly write the above equation as

$$F_B = -\rho g_\eta\left(\frac{4}{3}\pi R^3\right)$$

in which

$$g_\eta = -\frac{1}{5}\sin\alpha - g\cos\alpha$$

Thus

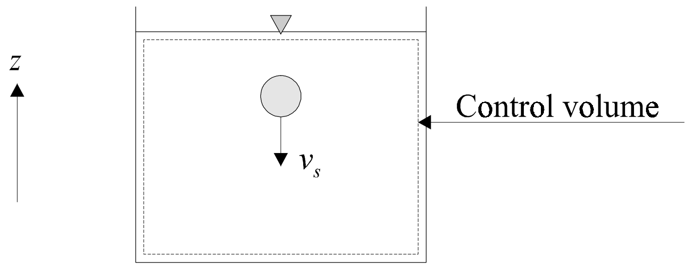
$$F_B = \rho g\left(\frac{4}{3}\pi R^3\right)\left(\frac{1}{5}\sin\alpha + \cos\alpha\right)$$

Problem 3.5

A 10 kg solid sphere at specific gravity $G = 2.65$ is submerged in a cubic meter of water. if the base of the container is 1 square meter, use the equation of momentum (3.27c) to determine the force on the bottom of the container when:

A) the solid sphere is released from rest and accelerates downward; and

B) the solid sphere settles at a constant velocity.



Solution The integral momentum equation in the z direction is

$$\begin{aligned} &\frac{d}{dt}\int_V \rho_m v_z dV + \int_A \rho_m \cancel{v_z}\left(v_x\frac{\partial x}{\partial n} + v_y\frac{\partial y}{\partial n} + v_z\frac{\partial z}{\partial n}\right)dA \\ =\; &\int_V \rho_m g_z dV - \int_A p\frac{\partial z}{\partial n}dA + \int_A \left(\cancel{\tau_{xz}}\frac{\partial x}{\partial n} + \cancel{\tau_{yz}}\frac{\partial y}{\partial n} + \cancel{\tau_{zz}}\frac{\partial z}{\partial n}\right)dA \end{aligned}$$

Since the velocities and shear stresses on the surface of the control volume are zeros, we have

$$\frac{d}{dt}\int_V \rho_m v_z dV = \int_V \rho_m g_z dV - \int_A p\frac{\partial z}{\partial n}dA$$

When the sphere falls down, there is an equivalent volume of water rising up, and the remaining water is rest. Therefore, we have

$$\begin{aligned} \frac{d}{dt}\int_V \rho_m v_z dV &= \frac{\partial}{\partial t}\int_{V_T-2V_s} \rho_w \cancel{v_z} dV + \frac{d}{dt}\int_{V_s} \rho_w v_z dV + \frac{d}{dt}\int_{V_s} \rho_s v_z dV \\ &= \rho_w \int_{V_s} \frac{dv_s}{dt}dV - \rho_s \int_{V_s} \frac{dv_s}{dt}dV \\ &= -\left(\rho_s - \rho_w\right) V_s \frac{dv_s}{dt} \end{aligned}$$

in which v_s is the sphere falling velocity.

The gravity term becomes

$$\begin{aligned} \int_V \rho_m g_z dV &= -\int_{V_T-V_s} \rho_w g dV - \int_{V_s} \rho_s g dV = -\rho_w g\left(V_T - V_s\right) - \rho_s g V_s \\ &= -\rho_w g V_T - \left(\rho_s - \rho_w\right) g V_s \end{aligned}$$

The pressure term becomes

$$-\int_A p\frac{\partial z}{\partial n}dA = -p\left(-1\right)A = pA = F_{\text{bottom}}$$

Finally, we have

$$\begin{aligned} F_{\text{bottom}} &= -(\rho_s-\rho_w)V_s\frac{dv_s}{dt}+\rho_w gV_T+(\rho_s-\rho_w)gV_s \\ &= -\left(1-\frac{\rho_w}{\rho_s}\right)M_s\frac{dv_s}{dt}+\rho_w gV_T+\left(1-\frac{\rho_w}{\rho_s}\right)M_s g \\ &= \left(1-\frac{\rho_w}{\rho_s}\right)\left(1-\frac{1}{g}\frac{dv_s}{dt}\right)M_s g+\rho_w gV_T \end{aligned} \tag{3.22}$$

in which $\rho_s/\rho_w = G = 2.65$, $g = 9.81$ m/s^2, $M_s = 10$ kg, $\rho_w = 1000$ kg/m^3, and $V_T = 1$ m^3.

For Case A, applying Newton's second law to the sphere gives that

$$\rho_s\left(\frac{4\pi}{3}R^3\right)\left(\frac{dv_s}{dt}\right)=(\rho_s-\rho_w)g\left(\frac{4\pi}{3}R^3\right)$$

then

$$\frac{1}{g}\frac{dv_s}{dt}=1-\frac{1}{G}$$

Substituting the above relation into (3.22) gives that

$$\begin{aligned} F_{\text{bottom}} &= \left(1-\frac{1}{G}\right)\left(\frac{1}{G}\right)M_s g+\rho_w V_T g \\ &= \left(1-\frac{1}{2.65}\right)\left(\frac{1}{2.65}\right)(10)(9.81)+(1000)(1)(9.81) \\ &= 9833\ \text{N} \end{aligned}$$

For case B, since

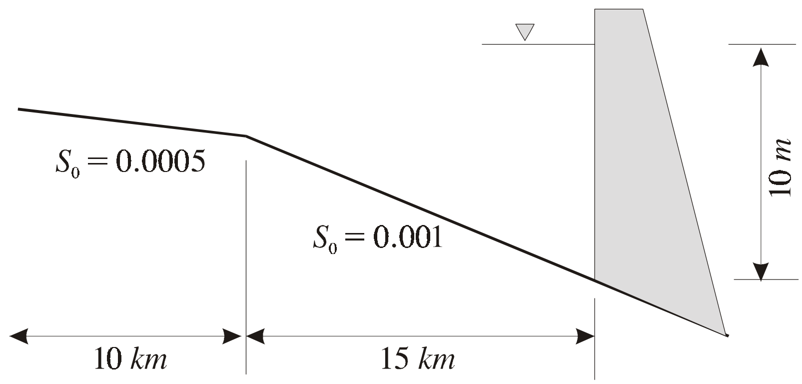
$$\frac{dv_s}{dt}=0$$

then

$$\begin{aligned} F_{\text{bottom}} &= \left(1-\frac{\rho_w}{\rho_s}\right)M_s g+\rho_w gV_T \\ &= \left(1-\frac{1}{2.65}\right)(10)(9.81)+(1000)(9.81)(1) \\ &= 9871\ \text{N} \end{aligned}$$

Computer Problem 3.1

Consider steady flow ($q = 3.72$ m^2/s) in the following impervious rigid boundary channel:



Assume that the channel width remains large and constant regardless of flow depth, and $f = 0.03$. Determine the distribution of the following parameters along the 25-km reach of the channel when the water surface elevation at the dam is 10 m above the bed elevation: (a) flow depth in m; (b) mean flow velocity in m/s; and (c) bed shear stress in N/m^2.

Solution (1) **Problem formulation** (the equation of backwater curve and boundary condition):

$$\frac{dh}{dx} = -\frac{S_0 - S_f}{1 - \mathrm{Fr}^2} \tag{3.23}$$

in which h =flow depth; x =distance from the dam; the negative sign "-" means the direction of x toward upstream; S_0 =bed slope; S_f = energy slope; and Fr=Froude number.

If Chezy resistance formula is used, the above equation can be rewritten as

$$\frac{dh}{dx} = -S_0 \frac{1 - \left(\frac{h_n}{h}\right)^3}{1 - \left(\frac{h_c}{h}\right)^3} \tag{3.24}$$

in which h_n =normal flow depth; and h_c =critical flow depth. h_n and h_c can be estimated by

$$h_n = \left(\frac{fq^2}{8gS_0}\right)^{\frac{1}{3}} \tag{3.25}$$

$$h_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \tag{3.26}$$

in which f =Darcy-Weishach friction factor; and g =gravitational acceleration.

The boundary condition is

$$h = 10 \text{ m} \qquad \text{at } x = 0 \text{ m} \tag{3.27}$$

(2) **Calculations of flow depth, velocity and bed shear stress**: A MatLab program has been written to handle the above equation and its boundary condition,

see Appendix I. After h is solved, the velocities and bed shear stresses are estimated from the following:

$$V = \frac{q}{h}, \qquad \tau = \frac{f}{8}\rho V^2$$

The results of the calculations are shown in the appendix II.

(3) **Discussions**: The normal depths in the two reaches can be calculated by solving the following two equations:

$$\frac{V}{\sqrt{ghS_0}} = \sqrt{\frac{8}{f}}$$

$$q = Vh$$

which gives that

$$h = \left(\frac{q^2 f}{8gS_0}\right)^{\frac{1}{3}}$$

Substituting the given values, we get for the reach near the dam

$$h_n = 1.7424 \text{ m}$$

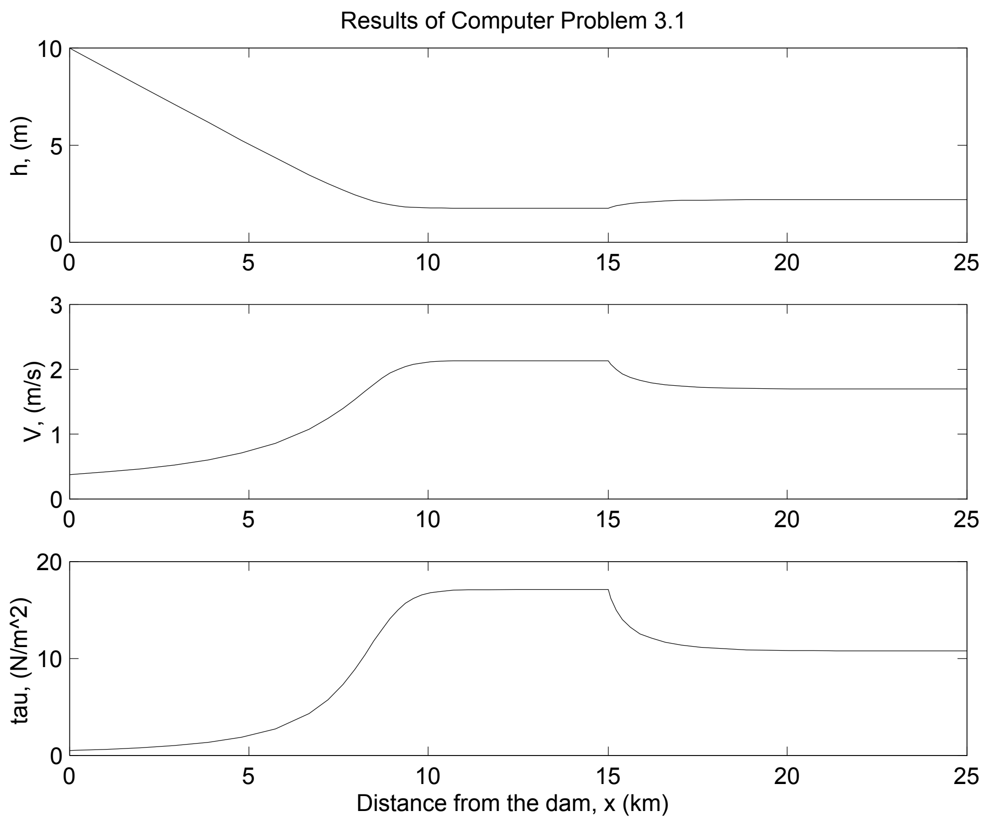
for the reach away from the dam

$$h_n = 2.1953 \text{ m}$$

The critical depth is

$$h_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} = \left(\frac{3.72^2}{9.81}\right)^{\frac{1}{3}} = 1.12 \text{ m}$$

Thus, for the reach near the dam, we have $h_c < h_n < h$, so the water surface profile is **M1**. For the reach away from the dam, $h_c < h < h_n$, so the water surface profile is **M2**, see the following plot. (continued)



Appendix I: MatLab Program

```
%Computer Problem 3.1

%General parameter
q = 3.72; %Unit discharge (m^2/s)
f = 0.03; %Darcy-Weisbach fractor
rho = 1000; %Water density

%Initial condition
x0 = 0; h0 = 10; xfinal = 15000;

S0 = 0.001;
[x,h1] = ode23v('cp3_1f',x0,xfinal,h0,S0,0);
v1 = q./h1; tau1 = f./8.*rho.*v1.^2;

table1 = [x'; h1'; v1'; tau1']';

%%%%%%%%%%%%%%%%%% Second segment %%%%%%%%%%%%%%%%%%
%Initial condition
```

```
x0 = 15000; h0 = h1(29); xfinal = 25000;

S0 = 0.0005;
[x,h2] = ode23v('cp3_1f',x0,xfinal,h0,S0,0);
v2 = q./h2; tau2 = f./8.*rho.*v2.^2;

fid = fopen('cp3_1.txt','w')
fprintf(fid,'%10.4f %10.4f %10.4f %10.4f\n',table')

%%%%%%%%%%%%%%%%% Plot Results %%%%%%%%%%%%%%%%%%%%
load cp3_1.txt, A = cp3_1;

subplot(311),plot(A(:,1)/1000,A(:,2))
ylabel('h, (m)'), title('Results of Computer Problem 3.1')

subplot(312),plot(A(:,1)/1000,A(:,3))
ylabel('V, (m/s)')

subplot(313),plot(A(:,1)/1000,A(:,4))
ylabel('tau, (N/m^2)'), xlabel('Distance from the dam, x (km)')

print -dps cp3_1.ps

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function dh = profile(x,h,S0,SS)
q = 3.72;
f = 0.03;
g = 9.81;
hn = (f.*q.^2./8./g./S0).^(1./3);
hc = (q.^2./g).^(1./3);
dh = -S0.*(1-(hn./h).^3)./(1-(hc./h).^3);
```

Appendix II: Results of Calculations

Distance from dam x (m)	Flow depth h (m)	Flow velocity V (m/s)	Bed shear tau (N/m^2)
0.0000	10.0000	0.3720	0.5189
117.1875	9.8833	0.3764	0.5313
1054.6875	8.9502	0.4156	0.6478
1992.1875	8.0187	0.4639	0.8071

2929.6875	7.0897	0.5247	1.0324
3867.1875	6.1649	0.6034	1.3654
4804.6875	5.2474	0.7089	1.8847
5742.1875	4.3439	0.8564	2.7501
6679.6875	3.4708	1.0718	4.3078
7204.0259	3.0097	1.2360	5.7290
7619.8459	2.6698	1.3934	7.2807
7955.4242	2.4213	1.5364	8.8518
8237.8084	2.2372	1.6628	10.3687
8485.2242	2.0996	1.7717	11.7714
8710.7659	1.9965	1.8632	13.0186
8938.5075	1.9150	1.9426	14.1512
9152.2674	1.8581	2.0021	15.0312
9362.8887	1.8181	2.0461	15.6995
9579.0843	1.7902	2.0780	16.1926
9809.2586	1.7711	2.1004	16.5436
10062.8775	1.7585	2.1155	16.7820
10352.0573	1.7506	2.1250	16.9341
10694.0808	1.7460	2.1306	17.0232
11116.3107	1.7436	2.1335	17.0692
11667.7685	1.7426	2.1347	17.0887
12454.7728	1.7424	2.1350	17.0938
13392.2728	1.7424	2.1349	17.0925
14329.7728	1.7424	2.1350	17.0933
15000.0000	1.7424	2.1350	17.0930
15000.0000	1.7424	2.1350	17.0930
15078.1250	1.7906	2.0775	16.1855
15225.6386	1.8612	1.9987	14.9809
15403.9984	1.9241	1.9334	14.0177
15624.5072	1.9811	1.8778	13.2223
15892.4813	2.0312	1.8314	12.5780
16214.7976	2.0740	1.7937	12.0648
16599.3329	2.1092	1.7637	11.6650
17055.8019	2.1371	1.7406	11.3618
17596.9071	2.1583	1.7236	11.1399
18221.9071	2.1732	1.7118	10.9882
18846.9071	2.1820	1.7049	10.8998
19471.9071	2.1872	1.7008	10.8474
20096.9071	2.1904	1.6983	10.8160
20721.9071	2.1923	1.6968	10.7971
21346.9071	2.1935	1.6959	10.7856
21971.9071	2.1942	1.6954	10.7787
22596.9071	2.1946	1.6950	10.7745

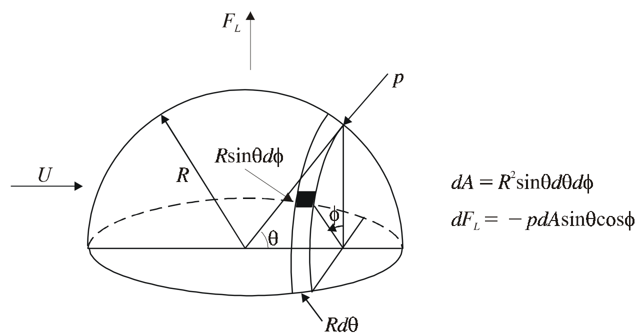
23221.9071	2.1949	1.6948	10.7719
23846.9071	2.1950	1.6947	10.7703
24471.9071	2.1951	1.6946	10.7694
25000.0000	2.1952	1.6946	10.7689

Chapter 4

Particle motions in inviscid fluids

Exercise

1. Substitute the relationship of pressure p from Equation (4.19) into Equation (E4.2.2) and solve for F_L.



Solution According to equation (4.19), the pressure distribution on the surface is

$$p = p_\infty + \frac{1}{2}\rho_m u_\infty^2 \left(1 - \frac{9}{4}\sin^2\theta\right)$$

Assume the pressure on the plate is p_∞. The lift force on the half-sphere is

$$\begin{aligned}
F_L &= \int_A (p_\infty - p)\sin\theta\cos\phi dA \\
&= -\frac{1}{2}\rho_m u_\infty^2 R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi d\phi \int_0^\pi \left(1 - \frac{9}{4}\sin^2\theta\right)\sin^2\theta d\theta \\
&= -\frac{1}{2}\rho_m u_\infty^2 R^2 (2)\left[\int_0^\pi \sin^2\theta d\theta - \frac{9}{4}\int_0^\pi \sin^4\theta d\theta\right] \\
&= -\rho_m u_\infty^2 R^2 \left[\frac{\pi}{2} - \frac{9}{4}\cdot\frac{3\pi}{8}\right] \\
&= \frac{11\pi}{32}\rho_m u_\infty^2 R^2
\end{aligned}$$

Problem 4.1

Determine the equation of pressure around a vertical half-cylinder from the Bernoulli equation [Eq. (3.21)] in a horizontal plane assuming $p_r = p_\infty = 0$ at $r = \infty$, where $v_r = u_\infty$.

Solution Eq. (3.21) is

$$p = \frac{\rho_m}{2}(v_r^2 - v^2) + p_r$$

Substituting $p_r = p_\infty = 0$ at $r = \infty$, where $v_r = u_\infty$, into the above equation, we have

$$p = \frac{\rho_m}{2}(u_\infty^2 - v^2) \tag{4.1}$$

From (4.9a) and (4.9b), we have the velocities around the cylinder ($r = R$) is

$$\begin{aligned} v_r &= 0 \\ v_\theta &= -2u_\infty \sin\theta + \frac{\Gamma_v}{2\pi R} \end{aligned}$$

which means that

$$v = v_\theta = -2u_\infty \sin\theta + \frac{\Gamma_v}{2\pi R}$$

Substituting the above into (4.1) gives that

$$\boxed{p = \frac{\rho_m}{2}\left[u_\infty^2 - \left(-2u_\infty \sin\theta + \frac{\Gamma_v}{2\pi R}\right)^2\right]}$$

Problem 4.2

Calculate the lift force in lb on a 4-m-diameter hemispherical tent under a 100-km/h wind.

Solution From (E4.2.2) or Exercise 4.1, we have

$$F_L = \frac{11\pi}{32}\rho_{\text{air}} u_\infty^2 R^2$$

Assume that $\rho_{\text{air}} = 1.2$ kg/m^3. Given $R = 2$ m, and

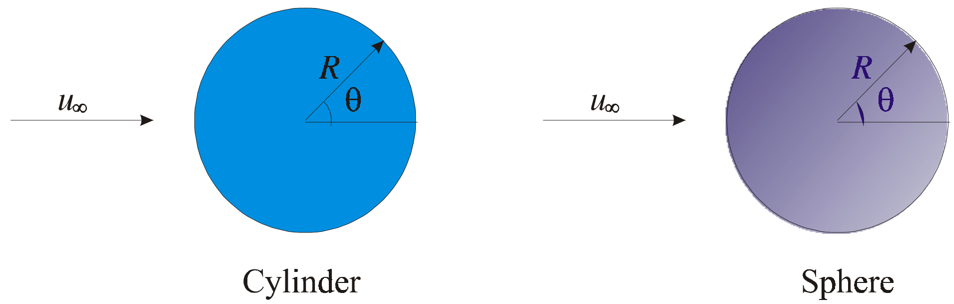
$$u_\infty = 100\frac{\text{km}}{\text{hr}}\left(\frac{1000 \text{ m}}{\text{km}}\right)\left(\frac{\text{hr}}{3600 \text{ s}}\right) = 27.78 \text{ m/s}$$

we have

$$\begin{aligned} F_L &= \frac{11\pi}{32}(1.2)(27.78)^2(2)^2 = 4000 \text{ N} \\ &= 4000\text{N}\left(\frac{\text{lb}}{4.448 \text{ N}}\right) = \boxed{899 \text{ lb}} \end{aligned}$$

Problem 4.3

Plot and compare the distribution of surface velocity, pressure, and boundary shear stress for irrotational flow without circulation around a cylinder and a sphere of radius R.



Solution (a) For a cylinder, the velocity distribution is

$$v_r = u_\infty \left(1 - \frac{R^2}{r^2}\right) \cos\theta$$
$$v_\theta = -u_\infty \left(1 + \frac{R^2}{r^2}\right) \sin\theta$$

On the cylinder surface at $r = R$, we have

$$v_r = 0$$
$$v_\theta = -2u_\infty \sin\theta$$

According Bernoulli equation, we have the pressure distribution on the surface is

$$p - p_\infty = \frac{\rho}{2} u_\infty^2 (1 - 4\sin^2\theta)$$

Assume that $p_\infty = 0$, then we have

θ	v_θ	p
$0°$	0	$\frac{1}{2}\rho u_\infty^2$
$90°$	$-2u_\infty$	$-\frac{3}{2}\rho u_\infty^2$
$180°$	0	$\frac{1}{2}\rho u_\infty^2$
$270°$	$2u_\infty$	$-\frac{3}{2}\rho u_\infty^2$

(b) For a sphere, the velocity distribution is

$$v_r = u_\infty \left(1 - \frac{R^3}{r^3}\right) \cos\theta$$
$$v_\theta = -u_\infty \left(1 + \frac{1}{2}\frac{R^3}{r^3}\right) \sin\theta$$

On the surface at $r = R$, we have the velocity distribution as

$$\begin{aligned} v_r &= 0 \\ v_\theta &= -\frac{3}{2} u_\infty \sin\theta \end{aligned}$$

and the pressure distribution as

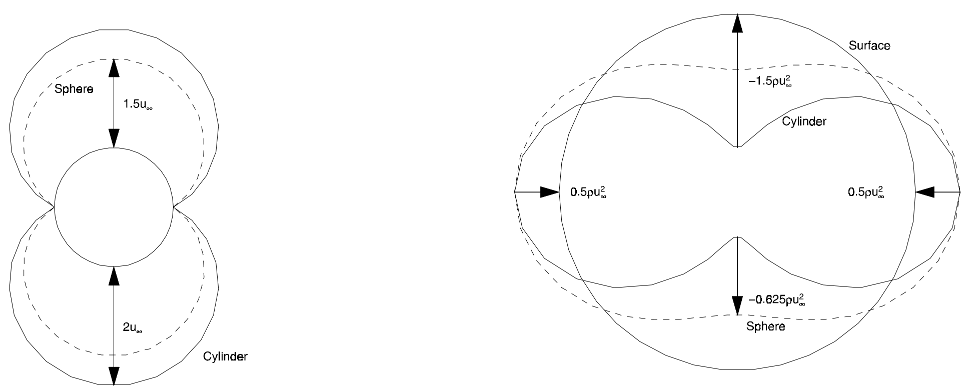
$$p - p_\infty = \frac{1}{2}\rho u_\infty^2 \left(1 - \frac{9}{4}\sin^2\theta\right)$$

Let $p_\infty = 0$, we have

θ	v_θ	p
$0°$	0	$\frac{1}{2}\rho u_\infty^2$
$90°$	$-\frac{3}{2}u_\infty$	$-\frac{5}{8}\rho u_\infty^2$
$180°$	0	$\frac{1}{2}\rho u_\infty^2$
$270°$	$\frac{3}{2}u_\infty$	$-\frac{5}{8}\rho u_\infty^2$

For potential flows, since we assume the fluid is inviscid, the shear stress everywhere is zero.

The comparisons of surface velocity distribution and pressure distribution are shown in the following two figures.



(a) Comparison of the velocity distributions (b) Comparison of prssure distributions

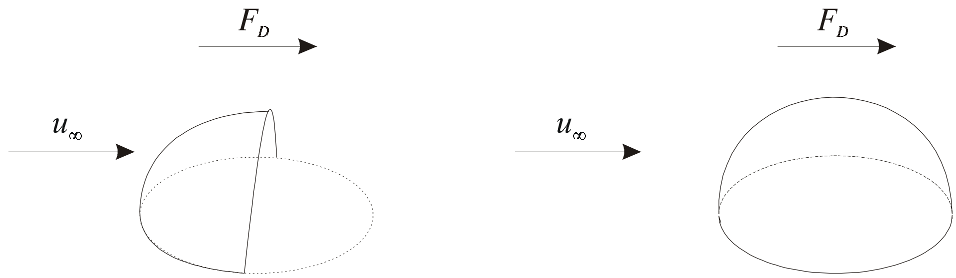
Problem 4.4

Calculate the drag force on the outside surface of

(a) a quarter-spherical shell (*hint*: neglect the pressure inside the shell)

$$F_D = \int_{-\pi/2}^{\pi/2}\int_{\pi/2}^{\pi} -pR^2 \sin\theta\cos\theta d\theta d\phi$$

(b) a half-spherical shell (*hint*: neglect the hydrostatic pressure distribution).



(*a*) (*b*)

Solution (a) Assume that $p_\infty = 0$, from Eq.(4.19) in the text, we have

$$p = \frac{\rho_m u_\infty^2}{2}\left(1 - \frac{9}{4}\sin^2\theta\right)$$

then

$$\begin{aligned} F_D &= \int_{-\pi/2}^{\pi/2}\int_{\pi/2}^{\pi} -pR^2 \sin\theta\cos\theta d\theta d\phi \\ &= -\frac{\rho_m u_\infty^2}{2}\pi R^2 \int_{\pi/2}^{\pi}\left(1 - \frac{9}{4}\sin^2\theta\right)\sin\theta\cos\theta d\theta \\ &= -\frac{\rho_m u_\infty^2}{2}\pi R^2 \left[\frac{1}{2}\sin^2\theta - \frac{9}{4}\cdot\frac{1}{4}\sin^4\theta\right]_{\pi/2}^{\pi} \\ &= -\frac{\rho_m u_\infty^2}{2}\pi R^2\left[-\frac{1}{2} + \frac{9}{16}\right] \\ &= -\frac{\rho_m u_\infty^2}{32}\pi R^2 \end{aligned}$$

(b) For the half-spherical shell, since the pressure distribution is symmetrical, see Figure (b) in Problem 4.3, in the wind side and the lee side, the drag force must be zero.

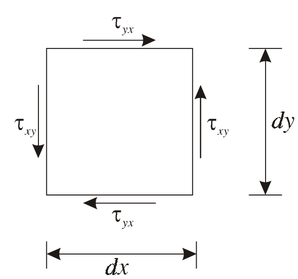
Chapter 5

Particle motions in Newtonian fluids

Exercises

1. Demonstrate that $\tau_{xy} = \tau_{yx}$ from the sum of moments about the center of an infinitesimal element.

   Solution Refer to the following figure.



   Taking the moments about the center of the infinitesimal element, when the element is in equilibrium state, we have

$$\tau_{xy} dy dx - \tau_{yx} dx dy = 0$$

   which gives that

$$\boxed{\tau_{xy} = \tau_{yx}}$$

2. Derive the x component of the Navier-Stokes equations in Table 5.1 from the equation of motion [Eq. (3.12a)] and the stress tensor components for incompressible Newtonian fluids [Eqs. (5.2) and (5.3)].

   Solution

$$a_x = g_x + \frac{1}{\rho}\frac{\partial \sigma_x}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} \tag{3.12a}$$

$$\sigma_x = -p + 2\mu\frac{\partial v_x}{\partial x} - \frac{2\mu}{3}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = -p + 2\mu\frac{\partial v_x}{\partial x}$$

in which $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$

$$\begin{aligned}
\tau_{yx} &= \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \\
\tau_{zx} &= \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} &= -\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 v_x}{\partial x^2} \\
\frac{\partial \tau_{yx}}{\partial y} &= \mu\frac{\partial^2 v_x}{\partial y^2} + \mu\frac{\partial^2 v_y}{\partial x \partial y} \\
\frac{\partial \tau_{zx}}{\partial z} &= \mu\frac{\partial^2 v_x}{\partial z^2} + \mu\frac{\partial^2 v_z}{\partial x \partial z}
\end{aligned}$$

thus,

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \\
&= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 v_x
\end{aligned}$$

Finally, we get

$$a_x = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 v_x$$

or

$$\boxed{a_x = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 v_x}$$

3. Cross-differentiate Equations (5.4a) and (5.4b) along y and x, respectively, to derive Equation (5.5c).

Solution

$$\begin{aligned}
\frac{\partial v_x}{\partial t} - v_y \otimes_z + v_z \otimes_y &= -\frac{g\partial H}{\partial x} + \nu_m \boldsymbol{\nabla}^2 v_x && (5.4a) \\
\frac{\partial v_y}{\partial t} - v_z \otimes_x + v_x \otimes_z &= -\frac{g\partial H}{\partial y} + \nu_m \boldsymbol{\nabla}^2 v_y && (5.4b)
\end{aligned}$$

Differentiating (5.4a) with respect to y gives

$$\frac{\partial^2 v_x}{\partial t \partial y} - \frac{\partial v_y}{\partial y}\otimes_z - v_y\frac{\partial \otimes_z}{\partial y} + \frac{\partial v_z}{\partial y}\otimes_y + v_z\frac{\partial \otimes_y}{\partial y} = -\frac{g\partial^2 H}{\partial x \partial y} + \nu_m \boldsymbol{\nabla}^2 \frac{\partial v_x}{\partial y} \qquad (5.1)$$

Differentiating (5.4b) with respect to x gives

$$\frac{\partial^2 v_y}{\partial t \partial x} - \frac{\partial v_z}{\partial x} \otimes_x - v_z \frac{\partial \otimes_x}{\partial x} + \frac{\partial v_x}{\partial x} \otimes_z + v_x \frac{\partial \otimes_z}{\partial x} = -\frac{g \partial^2 H}{\partial y \partial x} + \nu_m \boldsymbol{\nabla}^2 \frac{\partial v_y}{\partial x} \tag{5.2}$$

Subtracting (5.2) from (5.1) gives

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) + v_x \frac{\partial \otimes_z}{\partial x} + v_y \frac{\partial \otimes_z}{\partial y} - v_z \left(\frac{\partial \otimes_x}{\partial x} + \frac{\partial \otimes_y}{\partial y}\right) \\ & - \frac{\partial v_z}{\partial x} \otimes_x - \frac{\partial v_z}{\partial y} \otimes_y + \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \otimes_z \\ = \; & \nu_m \boldsymbol{\nabla}^2 \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \end{aligned} \tag{5.3}$$

Since

$$\begin{aligned} \otimes_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \\ -\frac{\partial v_z}{\partial z} &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \qquad \text{Mass continuity} \\ -\frac{\partial \otimes_z}{\partial z} &= \frac{\partial \otimes_x}{\partial x} + \frac{\partial \otimes_y}{\partial y} \quad \text{Vorticity continuity} \end{aligned}$$

thus, we have

$$\begin{aligned} & \frac{\partial \otimes_z}{\partial t} + v_x \frac{\partial \otimes_z}{\partial x} + v_y \frac{\partial \otimes_z}{\partial y} + v_z \frac{\partial \otimes_z}{\partial z} \\ = \; & \otimes_x \frac{\partial v_z}{\partial x} + \otimes_y \frac{\partial v_z}{\partial y} + \otimes_z \frac{\partial v_z}{\partial z} + \nu_m \boldsymbol{\nabla}^2 \otimes_z \end{aligned} \tag{5.5c}$$

4. Use the definition of stream function in Cartesian coordinates [Eq. (4.1)] to demonstrate that Equation (5.6) results from Equations (5.5).

   Solution For two-dimensional flows, we only have the vorticity in the z component, as shown in the above excercise (5.5c). Now,

$$\begin{aligned} \otimes_x &= \otimes_y = 0 \\ v_z &= 0 \\ \otimes_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \nabla^2 \Psi \end{aligned}$$

   then we have

$$\frac{d \nabla^2 \Psi}{dt} = \nu_m \nabla^2 \nabla^2 \Psi$$

   which, from $v_x = -\dfrac{\partial \Psi}{\partial y}$ and $v_y = \dfrac{\partial \Psi}{\partial x}$, gives

$$\boxed{\frac{\partial \nabla^2 \Psi}{\partial t} - \frac{\partial \Psi}{\partial y}\frac{\partial \nabla^2 \Psi}{\partial x} + \frac{\partial \Psi}{\partial x}\frac{\partial \nabla^2 \Psi}{\partial y} = \nu_m \nabla^4 \Psi}$$

5. Determine the shear stress component $\tau_{r\theta}$ in Equation (5.9c) from the tensor $\tau_{r\theta}$ in spherical coordinates (Table 5.3) and the velocity relationships [Eqs. (5.8a) and (5.8b)].

Solution The velocity distributions are

$$\begin{aligned} v_r &= u_\infty \left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta \\ v_\theta &= -u_\infty \left[1 - \frac{3}{4}\left(\frac{R}{r}\right) - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta \end{aligned}$$

The shear stress is

$$\begin{aligned} \tau_{r\theta} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right] \\ &= \mu u_\infty\left[-r\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r} - \frac{3}{4}\frac{R}{r^2} - \frac{1}{4}\frac{R^3}{r^4}\right) + \frac{1}{r}\left(1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\frac{R^3}{r^3}\right)\frac{\partial\cos\theta}{\partial\theta}\right] \\ &= \mu u_\infty\left[-r\sin\theta\left(-\frac{1}{r^2} + \frac{6}{4}\frac{R}{r^3} + \frac{R^3}{r^5}\right) - \sin\theta\left(\frac{1}{r} - \frac{3}{2}\frac{R}{r^2} + \frac{1}{2}\frac{R^3}{r^4}\right)\right] \\ &= \mu u_\infty\sin\theta\left(\frac{1}{r} - \frac{3}{2}\frac{R}{r^2} - \frac{R^3}{r^4} - \frac{1}{r} + \frac{3}{2}\frac{R}{r^2} - \frac{1}{2}\frac{R^3}{r^4}\right) \\ &= \mu u_\infty\sin\theta\left(-\frac{3}{2}\frac{R^3}{r^4}\right) = -\frac{3\mu u_\infty}{2R}\left(\frac{R}{r}\right)^4\sin\theta \end{aligned}$$

6. (a) Integrate the shear stress distribution [Eq.(5.9c)] to determine the surface drag in Equation (5.11) from Equation (5.10); and (b) integrate the dynamic pressure distribution [Eq.(5.9b)] to obtain the form drag in Equation (5.13) from Equation (5.12).

Solution The dynamic pressure and the shear stress distributions are

$$p_d = -\frac{3}{2}\frac{\mu_m u_\infty}{R}\left(\frac{R}{r}\right)^2\cos\theta \qquad (5.9b)$$

$$\tau = -\frac{3}{2}\frac{\mu_m u_\infty}{R}\left(\frac{R}{r}\right)^4\sin\theta \qquad (5.9c)$$

On the surface of the sphere,

$$p_d = -\frac{3}{2}\frac{\mu_m u_\infty}{R}\cos\theta \qquad (5.4)$$

$$\tau = -\frac{3}{2}\frac{\mu_m u_\infty}{R}\sin\theta \qquad (5.5)$$

(a) **Surface drag**

$$
\begin{aligned}
F'_D &= \int_0^{2\pi}\int_0^{\pi} -\tau R^2 \sin^2\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{3}{2}\frac{\mu_m u_\infty}{R} \sin\theta R^2 \sin^2\theta \, d\theta \, d\phi \\
&= (2\pi)\frac{3}{2}\mu_m u_\infty R \int_0^{\pi} \sin^3\theta \, d\theta = 6\pi\mu_m u_\infty R \int_0^{\frac{\pi}{2}} \sin^3 d\theta \\
&= 6\pi\mu_m u_\infty R \left(\frac{2}{3}\right) = 4\pi\mu_m u_\infty R
\end{aligned}
$$

(b) **Form drag**

$$
\begin{aligned}
F''_D &= \int_0^{2\pi}\int_0^{\pi} -p_d R^2 \sin\theta \, \cos\theta d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \frac{3}{2}\frac{\mu_m u_\infty}{R} \cos\theta \, R^2 \sin\theta \, \cos\theta d\theta \, d\phi \\
&= (2\pi)\frac{3}{2}\mu_m u_\infty R \int_0^{\pi} \sin\theta \, \cos^2\theta \, d\theta = 3\pi\mu_m u_\infty R \left(-\frac{\cos^3\theta}{3}\right)_0^{\pi} \\
&= 3\pi\mu_m u_\infty R \left(\frac{2}{3}\right) = 2\pi\mu_m u_\infty R
\end{aligned}
$$

7. Derive Rubey's fall velocity equation [Eqs. (5.23a) and (5.23b)] combining Equations (5.18) and (5.22b).

Solution

$$
\begin{aligned}
\omega^2 &= \frac{4}{3C_D}(G-1)gd_s \\
C_D &= \frac{24}{\mathrm{Re}} + 2 \\
\mathrm{Re} &= \frac{\omega d_s}{\nu}
\end{aligned}
$$

$$\omega^2\left(\frac{24\nu}{\omega d_s} + 2\right) - \frac{4}{3}(G-1)gd_s = 0$$

$$2\omega^2 + \frac{24\nu}{d_s}\omega - \frac{4}{3}(G-1)gd_s = 0$$

Solving for ω gives that

$$\omega = \frac{1}{d_s}\left[\sqrt{\frac{2g}{3}(G-1)\,d_s^3 + 36\nu^2} - 6\nu\right]$$

or

$$\omega = \left[\sqrt{\frac{2}{3} + \frac{36\nu^2}{(G-1)\,gd_s^3}} - \sqrt{\frac{36\nu^2}{(G-1)\,gd_s^3}}\right]\sqrt{(G-1)\,gd_s}$$

8. Substitute the appropriate stress tensor components for the flow of Newtonian fluids in Cartesian coordinates [Eqs. (5.2) and (5.3] into the last four terms in parentheses of Equation (5.26) to obtain the energy dissipation function in Equation (5.27).

   Solution Consider the following relations.

$$\begin{aligned}
\tau_{xx} &= 2\mu_m \frac{\partial v_x}{\partial x} - \frac{2}{3}\mu_m \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \\
\tau_{yy} &= 2\mu_m \frac{\partial v_y}{\partial y} - \frac{2}{3}\mu_m \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \\
\tau_{zz} &= 2\mu_m \frac{\partial v_z}{\partial z} - \frac{2}{3}\mu_m \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \\
\tau_{zx} &= \mu_m \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right), \quad \tau_{zy} = \mu_m \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right), \quad \tau_{xy} = \mu_m \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)
\end{aligned}$$

   Then we have

$$\begin{aligned}
\chi_D &= \left(\tau_{xx} \frac{\partial v_x}{\partial x} + \tau_{yy} \frac{\partial v_y}{\partial y} + \tau_{zz} \frac{\partial v_z}{\partial z} \right) \\
&\quad + \tau_{zx} \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) + \tau_{zy} \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) + \tau_{xy} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \\
&= 2\mu_m \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] - \frac{2\mu_m}{3} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)^2 \\
&\quad + \mu_m \left[\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)^2 + \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 \right]
\end{aligned}$$

   Note: The term $-p \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$ in Equation (5.27) is not a dissipation term.

9. Describe each member of the dissipation function [Eq. (5.28)] in terms of the fundamental modes of deformation shown in Figure 3.2 (translation, linear deformation, angular deformation, and rotation of a fluid element).

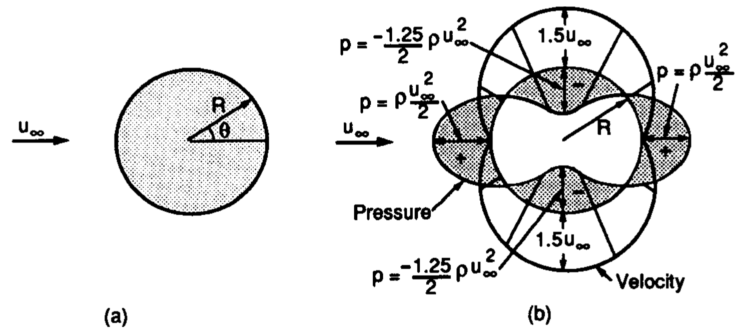
   Solution

$$\begin{aligned}
\chi_D &= 2\mu_m \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] \\
&\quad + \mu_m \left[\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)^2 + \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 \right]
\end{aligned}$$

   in which the terms in the first bracket [ ] are due to linear deformation, and those in the second bracket [ ] are due to angular deformation.

Problem 5.1

Plot the velocity, dynamic pressure, and shear stress distributions around the surface of a sphere for creeping motion given by Stokes' law [Eqs. (5.8) and (5.9)] and compare with irrotational flow without circulation (Problem 4.3).



Solution (a) For creeping flow, the velocity is zero everywhere on the surface because of no-slipping condition. The pressure distribution on the surface is

$$p = p_0 - \rho g R \cos\theta - \frac{3}{2}\frac{\mu u_\infty}{R}\cos\theta$$

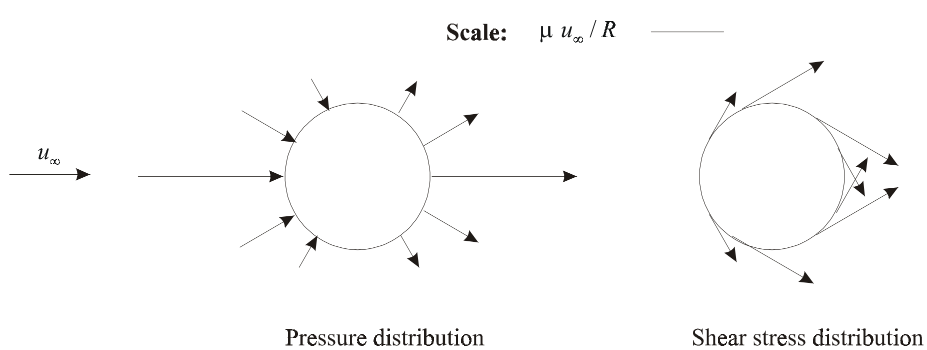
in which the first term is the ambient pressure, the second term is the static pressure, and the third term is the dynamic pressure whose distribution on the surface is then

$$p_d = -\frac{3}{2}\frac{\mu u_\infty}{R}\cos\theta$$

The shear stress distribution on the surface is

$$\tau_{r\theta} = -\frac{3}{2}\frac{\mu u_\infty}{R}\sin\theta$$

The plots are shown below.



θ	$p_d/(\mu u_\infty/R)$	$\tau_{r\theta}/(\mu u_\infty/R)$
0	-1.5	0
30°	-1.3	-0.75
60°	-0.75	-1.3
90°	0	-1.5
120°	0.75	-1.3
150°	1.3	-0.75
180°	1.5	0
210°	1.3	0.75
240°	0.75	1.3
270°	0	1.5
300°	-0.75	1.3
330°	-1.3	0.75
360°	-1.5	0

Some typical values are shown in the above table.

(b) For irrotational flow, because the fluid is inviscid, there is no shear stress on the surface. The velocity and pressure distributions have been shown in Figure (b) of the problem.

The comparison between Stokes' flow and irrotational flow at the surface of the sphere is as follows:

	Velocity	Pressure	Shear stress
Stokes' flow	zero	due to velocity change and viscosity	non-zero
Irrotational flow	non-zero	due to velocity change	zero

Problem 5.2

Plot Rubey's relationship for the drag coefficient C_D in Figure 5.2. How does it compare with the experimental measurements? At a given Re_p, which of Equations (5.18) and (5.19) induces larger settling velocities?

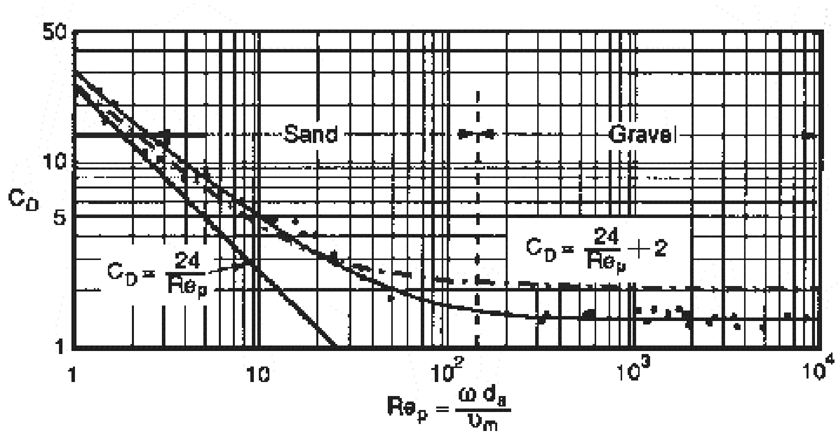
Solution Rubey's drag coefficient equation:

$$C_D = \frac{24}{\mathrm{Re}_p} + 2$$

The typical values are listed in the following table. .

Re_p	2	5	10	50	100	500	1000	∞
C_D	14	6.8	4.4	2.48	2.24	2.048	2.024	2.0

It is compared with Figure 5.2 as follows, in which the dashed line denotes Rubey's equation.



Rubey's equation for the drag coefficient, C_D, underpredicts the value of C_D for $\mathrm{Re}_p < 20$, and overpredicts the value of C_D for $\mathrm{Re}_p > 20$. For a given Reynolds number, Rubey's equation always underpredicts the value of fall velocity compared to Eq. (5.19), i.e., $C_D = \dfrac{24}{\mathrm{Re}_p} + 1.5$.

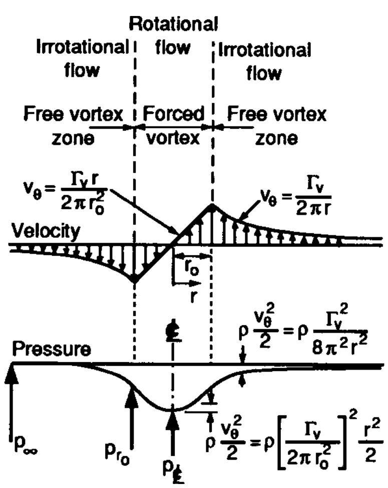
Problem 5.3

Evaluate the dissipation function χ_D from Table 5.5 for a vertical axis Rankine vortex described in cylindrical coordinates by (a) forced vortex

$$v_\theta = \frac{\Gamma_v r}{2\pi r_0^2}, \qquad v_z = v_r = 0 \qquad \text{(rotational flow for } r < r_0\text{)}$$

and (b) free vortex

$$v_\theta = \frac{\Gamma_v}{2\pi r}, \qquad v_z = v_r = 0 \qquad \text{(irrotational flow for } r > r_0\text{)}$$



Solution

$$\begin{aligned}\chi_D &= \mu_m \left\{ 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2 \right] \right. \\ &\left. + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]^2 \right\}\end{aligned}$$

For $v_\theta = f(r)$, and $v_z = v_r = 0$, we have

$$\begin{aligned}\chi_D &= \mu_m \left\{ 2\left[\left(\cancel{\frac{\partial v_r}{\partial r}}\right)^2 + \left(\frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r}{r}}\right)^2 + \left(\cancel{\frac{\partial v_z}{\partial z}}\right)^2 \right] \right. \\ &\left. + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}}\right]^2 + \left[\frac{1}{r}\cancel{\frac{\partial v_z}{\partial \theta}} + \cancel{\frac{\partial v_\theta}{\partial z}}\right]^2 + \left[\cancel{\frac{\partial v_r}{\partial z}} + \cancel{\frac{\partial v_z}{\partial r}}\right]^2 \right\} \\ &= \mu_m \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2\end{aligned}$$

(a) Forced vortex:

$$\chi_D = \mu_m \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2 = \mu_m \left[r\frac{\partial}{\partial r}\left(\frac{\Gamma_v}{2\pi r_0^2}\right)\right]^2 = 0$$

(b) Free vortex:

$$\chi_D = \mu_m \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2 = \mu_m \left[r\frac{\partial}{\partial r}\left(\frac{\Gamma_v}{2\pi r^2}\right)\right]^2 = \frac{\mu_m \Gamma_v^2}{\pi^2 r^4}$$

Problem 5.4

The sediment size distribution of a 1,200-mg sample is to be determined using the BWT. If the water temperature is 24°C and the solid weight for each 10-ml withdrawal is given, complete the following table and deter- mine the particle size distribution:

Particle diameter (mm)	Withdrawal time (min)	Sample volume (ml)	Dry weight of sediment (mg)	Cumulative dry weight (mg)	Percent settled	Percent finer[a] (%W)
0.25	0.485	10	144	144	12	
0.125		10	72	216	18	
0.0625		10	204	420	35	78
0.0312		10	264	684	57	58
0.0156		10	252			
0.0078		10	84		84	
0.0039		10	48	1068		
0.00195	4,461	10	45			

[a]See Figure 5.5

Solution The withdrawal time can be found from Figure 5.7 (p.84).

Particle diameter (mm)	Withdrawal time (min)	Sample volume (ml)	Dry weight of sediment (mg)	Cumulative dry weight (mg)	Percent settled	Percent finer[a] (%W)
0.25	0.485	10	144	144	12	**94**
0.125	**1.32**	10	72	216	18	**90**
0.0625	**4.33**	10	204	420	35	78
0.0312	**17.4**	10	264	684	57	58
0.0156	**69.6**	10	252	**936**	**78**	**48**
0.0078	**279**	10	84	**1020**	84	**30**
0.0039	**1114**	10	48	1068	**89**	**14**
0.00195	4,461	10	45	**1113**	**99**	**11**

[a] These values are obtained from Figure 5.5.

Computer Problem 5.1

Write a simple computer program to determine the particle size d_s, the fall velocity ω, the flocculated fall velocity ω_f, the particle Reynolds number Re_p, the dimensionless particle diameter d_*, and the time of settling per meter of quienscent water at 5°C, and complete the following table:

Class name	d_s (mm)	ω (cm/s)	ω_f (cm/s)	Re_p	d_*	Settling time
Medium clay						
Medium silt						
Medium sand						
Medium gravel						
Small cobble						
Medium boulder						

Solution Basic equations are

$$
\begin{aligned}
d_* &= d_s \left[\frac{(G-1)g}{\nu_m^2}\right]^{\frac{1}{3}} \\
\omega &= \frac{8\nu_m}{d_s}\left[\left(1+0.0139 d_*^3\right)^{0.5}-1\right] \\
\omega_f &= \frac{250}{d_s^2}\omega \qquad \text{if } d_s < 40\ \mu\text{m}
\end{aligned}
$$

in which d_s is in micrometers.

$$t = \begin{cases} L/\omega & \text{if flocculation does not ocuur} \\ L/\omega_f & \text{if flocculation occurs} \end{cases}$$

$$\text{Re}_p = \begin{cases} \omega d_s/\nu_m & \text{if flocculation does not ocuur} \\ \omega_f d_s/\nu_m & \text{if flocculation occurs} \end{cases}$$

The results are shown in the following table.

Class name	d_s (mm)	ω (cm/s)	ω_f^{a} (cm/s)	Re_p	d_*	Settling t^{b} (s)	Settling t_f^{c} (s)
	1020	384		2.59×10^6	19600	0.260	
Medium boulder	512	272		9.16×10^5	9810	0.368	
	128	136		1.14×10^5	2450	0.737	
Small cobble	64	96		4.05×10^4	1230	1.04	
	16	47.9		5050	306	2.09	
Medium gravel	8	33.8		178	153	2.96	
	0.5	6.4		21.1	9.58	15.6	
Medium sand	0.25	2.86		4.71	4.79	34.9	
	0.031	0.0569		3.02×10^{-3}	0.594	1757	
Medium silt	0.016	0.0152		1.56×10^{-3}	0.306	6579	
	0.002	2.37×10^{-4}	0.0148	1.95×10^{-4}	0.0383		6750
Medium clay	0.001	5.93×10^{-5}	0.0148	9.76×10^{-5}	0.0192		6750

Notes: [a]Flocculation does not occur if $d_s \geq 4 \times 10^{-5}$ m.
[b]Settling time $t = L/\omega$ if without flocculation.
[c]$t_f = L/\omega_f$ if flocculation occurs.

MATLAB Program

```
%Program: Computer Program 5.1
G = 2.65; g = 9.81; T = 5; L = 1;

nu_m = (1.14-0.031.*(T-15)+0.00068.*(T-15).^2).*1e-6;
d_s = [1024 512 128 64 16 8 0.5 0.25 0.031 0.016 0.002 0.001].*1e-3;

d_star = d_s.*(((G-1).*g)./nu_m.^2).^(1./3);
omega = 8.*nu_m./d_s.*(sqrt(1+0.0139.*d_star.^3)-1);
omega_f = 250./(d_s.*1e6).^2.*omega;

for i=1:12
    if d_s(i) < 4e-5
       Re_p(i) = omega_f(i).*d_s(i)./nu_m; t(i) = L./omega_f(i);
    else
       Re_p(i) = omega(i).*d_s(i)./nu_m; t(i) = L./omega(i);
```

```
    end
end

table = [d_s;omega;omega_f;Re_p;d_star;t];
fid = fopen('cp5_1.txt','w');
fprintf(fid,'%11.2e& %11.2e& %11.2e& %11.2e& %11.2e& %11.2e\\\\\n',table);
fclose(fid);

type cp5_1.txt
```

Chapter 6

Turbulent velocity profiles

Exercises

1. Substitute Equations (6.la-d) into the Navier-Stokes equations (Table 5.1) to obtain Equation (6.4a).

   Solution Consider the equation in the x direction.

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = g_x - \frac{1}{\rho_m} \frac{\partial p}{\partial x} + \nu_m \nabla^2 v_x \tag{6.1}$$

Adding $v_x \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$, which is zero, to the left-hand side gives that

$$\frac{\partial v_x}{\partial t} + \frac{\partial v_x^2}{\partial x} + \frac{\partial v_x v_y}{\partial y} + \frac{\partial v_x v_z}{\partial z} = g_x - \frac{1}{\rho_m} \frac{\partial p}{\partial x} + \nu_m \nabla^2 v_x \tag{6.2}$$

Assuming that

$$\begin{aligned} p &= \overline{p} + p^+ \\ v_x &= \overline{v}_x + v_x^+ \\ v_y &= \overline{v}_y + v_y^+ \\ v_z &= \overline{v}_z + v_z^+ \end{aligned}$$

then (6.2) becomes that

$$\begin{aligned} & \frac{\partial \left(\overline{v}_x + v_x^+ \right)}{\partial t} + \frac{\partial \left(\overline{v}_x + v_x^+ \right)^2}{\partial x} + \frac{\partial \left(\overline{v}_x + v_x^+ \right) \left(\overline{v}_y + v_y^+ \right)}{\partial y} \\ & + \frac{\partial \left(\overline{v}_x + v_x^+ \right) \left(\overline{v}_z + v_z^+ \right)}{\partial z} \\ = \; & g_x - \frac{1}{\rho_m} \frac{\partial \left(\overline{p} + p^+ \right)}{\partial x} + \nu_m \nabla^2 \left(\overline{v}_x + v_x^+ \right) \end{aligned}$$

Taking Reynolds average over the above equation and considering

$$\overline{p^+} = \overline{v_x^+} = \overline{v_y^+} = \overline{v_z^+} = 0$$

and

$$\begin{aligned}
\overline{v_x^2} &= \overline{(\overline{v}_x + v_x^+)^2} = \overline{v}_x^2 + \overline{v_x'^2} \\
\overline{v_x v_y} &= \overline{v}_x \overline{v}_y + \overline{v_x^+ v_y^+} \\
\overline{v_x v_z} &= \overline{v}_x \overline{v}_z + \overline{v_x^+ v_z^+}
\end{aligned}$$

yields

$$\frac{\partial \overline{v}_x}{\partial t} + \overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} + \overline{v}_z \frac{\partial \overline{v}_x}{\partial z} + \left[\frac{\partial \overline{v_x^+ v_x^+}}{\partial x} + \frac{\partial \overline{v_x^+ v_y^+}}{\partial y} + \frac{\partial \overline{v_x^+ v_z^+}}{\partial z} \right] = g_x - \frac{1}{\rho_m} \frac{\partial \overline{p}}{\partial x} + \nu_m \nabla^2 \overline{v}_x$$

or

$$\frac{\partial \overline{v}_x}{\partial t} + \overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} + \overline{v}_z \frac{\partial \overline{v}_x}{\partial z} = g_x - \frac{1}{\rho_m} \frac{\partial \overline{p}}{\partial x} + \nu_m \nabla^2 \overline{v}_x - \left[\frac{\partial \overline{v_x^+ v_x^+}}{\partial x} + \frac{\partial \overline{v_x^+ v_y^+}}{\partial y} + \frac{\partial \overline{v_x^+ v_z^+}}{\partial z} \right]$$

Similarly, we can derive the equations in the y and z directions.

2. Demonstrate that Equation (6.13) is obtained from Equation (6.12) when $z_0 = k_s'/30$.

Solution

$$\begin{aligned}
\frac{v_x}{u_*} &= \frac{1}{\kappa} \ln \frac{z}{z_0} = \frac{1}{\kappa} \ln \frac{30z}{k_s'} \\
&= \frac{2.3}{0.4} \log \frac{30z}{k_s'} = 5.75 \log \frac{30z}{k_s'}
\end{aligned}$$

3. Demonstrate that Equation (6.20) is obtained from Equation (6.12) when $z_0 = \nu_m / 9u_*$.

Solution

$$\begin{aligned}
\frac{v_x}{u_*} &= \frac{1}{\kappa} \ln \frac{z}{z_0} = \frac{1}{\kappa} \ln \frac{9u_* z}{\nu_m} \\
&= \frac{2.3}{0.4} \log \frac{9u_* z}{\nu_m} \\
&= 5.75 \log \frac{u_* z}{\nu_m} + 5.75 \log 9 \\
&= 5.75 \log \frac{u_* z}{\nu_m} + 5.5
\end{aligned}$$

4. Derive Equation (6.23) from Equations (6.22b) and (6.20) at $z = \delta$.

   Solution Equating (6.22b) and (6.20) gives that

$$\frac{u_* \delta}{\nu_m} = 5.75 \log \frac{u_* \delta}{\nu_m} + 5.5$$

Solving for $\frac{u_* \delta}{\nu_m}$ gives that

$$\frac{u_* \delta}{\nu_m} = 11.6$$

or

$$\delta = \frac{11.6 \nu_m}{u_*}$$

Problem 6.1

Consider the clear-water and sediment-laden velocity profiles measured in a smooth laboratory flume at a constant discharge by Coleman (1986). Notice the changes in the velocity profiles due to the presence of sediments. Determine the von Karman constant κ from Equation (6.12) for the two velocity profiles in the following tabulation, given $u_* = 0.041$ m/s, $d_s = 0.105$ mm, $Q = 0.064$ m^3/s, $h \approx 0.17$ m, $S_f = 0.002$, and $W = 0.356$ m.

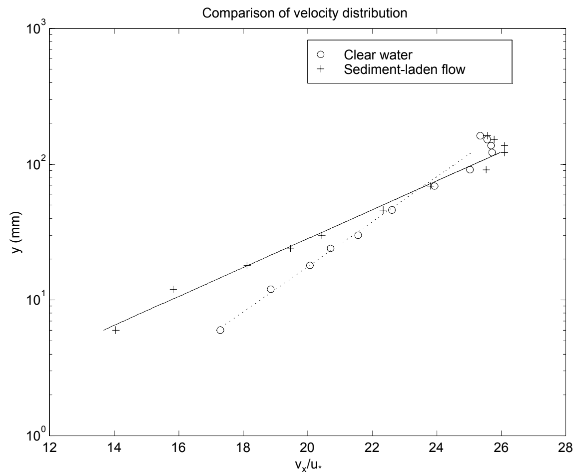
Elevation[a] (mm)	Clear-water flow velocity (m/s)	Sediment-laden velocity (m/s)	Concentration by volume
6	0.709	0.576	2.1×10^{-2}
12	0.773	0.649	1.2×10^{-2}
18	0.823	0.743	7.7×10^{-3}
24	0.849	0.798	5.9×10^{-3}
30	0.884	0.838	4.8×10^{-3}
46	0.927	0.916	3.2×10^{-3}
69	0.981	0.976	2.5×10^{-3}
91	1.026	1.047	1.6×10^{-3}
122	1.054	1.07	8.0×10^{-4}
137	1.053	1.07	
152	1.048	1.057	
162	1.039	1.048	

[a] Elevation above the bed.

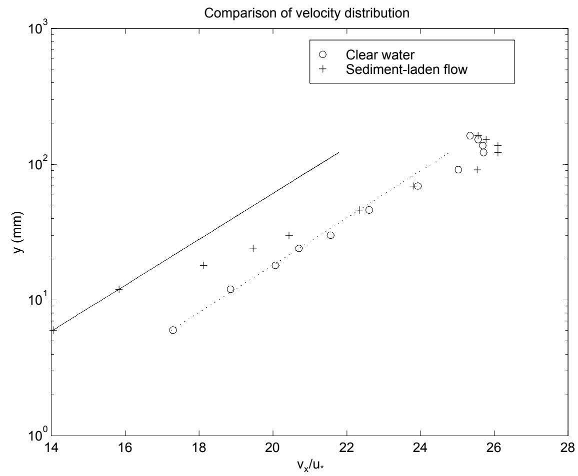
Solution From (6.12)

$$\frac{v_x}{u_*} = \frac{1}{\kappa} \ln \frac{z}{z_0} = \frac{1}{\kappa} \ln z + \text{constant}$$

A MATLAB Program has been written to plot the velocity distribution and estimate κ. If the main portions of both velocity distributions are considered, κ is calculated as 0.3822 for the clear water and 0.2451 for the sediment-laden flow. If the lowest portions of both velocity profiles are considered, $\kappa = 0.3999$ for the clear water and 0.3893 for the sediment-laden flow.



The main portion of the velocity is considered



The lowest portion is considered

Appendix: MATLAB Program

```
%Program: Problem 6.1, PYJ Page 109

u_star = 0.041;   %m/s
d_s = 0.105;   %mm
Q = 0.064;  %m^3/s
h = 0.17;  %m
S_f = 0.002; W = 0.356;  %m

y = [6 12 18 24 30 46 69 91 122 137 152 162];
u1 = [0.709 0.773 0.823 0.849 0.884 0.927 0.981 1.026 1.054...
      1.053 1.048 1.039]./u_star;
u2 = [0.576 0.649 0.743 0.798 0.838 0.916 0.976 1.047 1.07...
      1.07 1.057 1.048]./u_star;
semilogy(u1,y,'o',u2,y,'+'), hold on
legend('Clear water','Sediment-laden flow')

c1 = polyfit(log(y(1:5)),u1(1:5),1);
uu1 = polyval(c1,log(6:0.1:122));
semilogy(uu1,[6:0.1:122],':')

c2 = polyfit(log(y(1:5)),u2(1:5),1);
uu1 = polyval(c2,log(6:0.1:122));
semilogy(uu1,[6:0.1:122],'-')
```

```
sxlabel('v_x/u_*'), sylabel('y (mm)')
title('Comparison of velocity distribution')

k0 = 1./c1(1), km = 1./c2(1), hold off
```

Problem 6.2

(a) In turbulent flows, determine the elevation at which the local velocity v_x is equal to the depth-averaged velocity V_x. (b) Determine the elevation at which the local velocity v_x equals the shear velocity u_*.

Solution (a) Consider the general velocity distribution.

$$\frac{v_x}{u_*} = \frac{1}{\kappa}\ln z + C_0$$

The depth-averaged velocity is

$$\begin{aligned}\frac{V_x}{u_*} &= \frac{1}{h}\int_0^h \left[\frac{1}{\kappa}\ln z + C_0\right] dz = \frac{1}{h}\left[\frac{1}{\kappa}\left(z\ln z - z\right)\Big|_0^h + C_0 h\right] \\ &= \frac{1}{h}\left[\frac{1}{\kappa}\left(h\ln h - h\right) + C_0 h\right] = \frac{1}{\kappa}\left(\ln h - 1\right) + C_0\end{aligned}$$

Equating the above two equations gives that

$$\frac{1}{\kappa}\ln z = \frac{1}{\kappa}\left(\ln h - 1\right)$$

or

$$\ln\frac{h}{z} = 1$$

which gives that

$$\boxed{z = 0.368h}$$

(b) For hydraulically smooth wall, when $u = u_*$, $u/u_* = u_* z/\nu < 5$. Therefore, we need to use the liner law, i.e.

$$1 = \frac{u_* z}{\nu}$$

which gives that

$$\boxed{z = \frac{\nu_m}{u_*}}$$

For hydraulically rough wall, a point where $u = u_*$ does not exist.

Problem 6.3

Determine the Darcy-Weisbach friction factor f from the data in Problem 6.1.

Solution From Problem 6.1, we have

$$u_* = 0.041 \text{ m/s} \quad \text{and} \quad V = \frac{Q}{Wh} = \frac{0.064}{(0.356)(0.17)} = 1.058 \text{ m/s}$$

then

$$f = 8\left(\frac{u_*}{V}\right)^2 = 8\left(\frac{0.041}{1.058}\right)^2 = 0.012$$

Problem 6.4

(a) Calculate the laminar sublayer thickness δ in Problem 6.1. (b)Estimate the range of laminar sublayer thicknesses for bed slopes $10^{-5} < S_0 < 0.01$ and flow depths 0.5 m$< h < 5$ m.

Solution (a) Assume that the temperature is 20°C, then $\nu_m = 10^{-6}$ m^2/s. From (6.23) on page 100, we have

$$\delta = \frac{11.6\nu_m}{u_*} = \frac{11.6\left(10^{-6}\right)}{0.041} = 2.86 \times 10^{-4} \text{ m} = 0.286 \text{ mm}$$

(b) The lower limit of the shear velocity is

$$u_* = \sqrt{ghS} = \sqrt{9.81\,(0.5)\left(10^{-5}\right)} = 0.007 \text{ m/s}$$

The upper limit if the shear velocity is

$$u_* = \sqrt{ghS} = \sqrt{9.81\,(5)\,(0.01)} = 0.7 \text{ m/s}$$

So the range of δ is

$$\frac{11.6\left(10^{-6}\right)}{0.007} \text{ m} > \delta > \frac{11.6\left(10^{-6}\right)}{0.7} \text{ m}$$

i.e.

$$1.66 \text{ mm} > \delta > 0.0166 \text{ mm}$$

Problem 6.5

From turbulent velocity measurements at two elevations (v_1 at z_1, and v_2 at z_2) in a wide rectangular channel, use Equation (6.12) to determine the shear velocity u_*; the boundary shear stress τ_0; and the laminar sublayer thickness δ.

Solution Equation (6.12) is

$$\frac{v_x}{u_*} = \frac{1}{\kappa}\ln z - \frac{1}{\kappa}\ln z_0$$

- The shear velocity can be found as follows.

$$\frac{v_1}{u_*} = \frac{1}{\kappa}\ln z_1 - \frac{1}{\kappa}\ln z_0 \quad \text{and} \quad \frac{v_2}{u_*} = \frac{1}{\kappa}\ln z_2 - \frac{1}{\kappa}\ln z_0 \tag{6.4}$$

From the above two equations, we get

$$\frac{v_1 - v_2}{u_*} = \frac{1}{\kappa}(\ln z_1 - \ln z_2) = \frac{1}{\kappa}\ln\frac{z_1}{z_2}$$

i.e.

$$\boxed{u_* = \kappa\frac{(v_1 - v_2)}{\ln\dfrac{z_1}{z_2}}}$$

- The boundary shear stress can be found by

$$\tau_0 = \rho u_*^2 = \rho\kappa^2\left(\frac{v_1 - v_2}{\ln\dfrac{z_1}{z_2}}\right)^2$$

- The laminar sublayer is

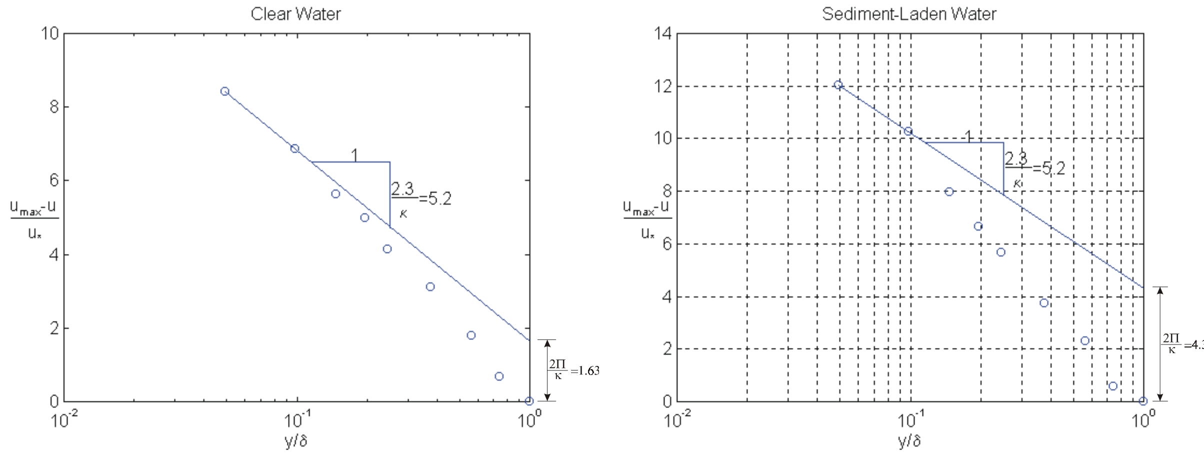
$$\delta = \frac{11.6\nu_m}{u_*} = 11.6\frac{\nu_m}{\kappa}\frac{\ln\dfrac{z_1}{z_2}}{(v_1 - v_2)}$$

Problem 6.6

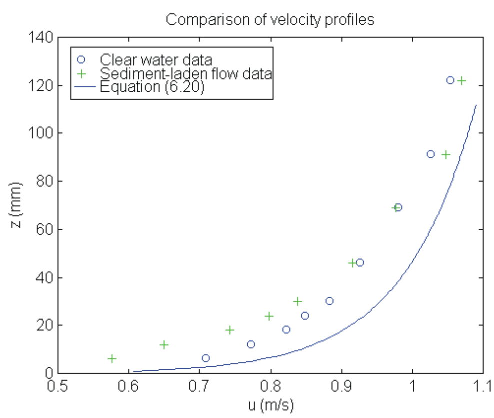
(a) With reference to Problem 6. 1, evaluate the parameters κ and Π_w from the velocity defect formulation in Equation (6.26). Compare the value of κ with the value obtained previously (Problem 6.1) from Equation (6.12). (b) Compare the experimental velocity profiles from Problem 6.1 with the velocity profiles calculated from Equations (6.13) and (6.20).

Solution (a) From the following plots, we get that

$$\begin{aligned}
\kappa &= \frac{2.3}{5.2} = 0.44 \text{ for both clear water and sediment-laden flows} \\
\Pi &= \frac{1.63\kappa}{2} = 0.34 \quad \text{for clear water} \\
\Pi &= \frac{4.3\kappa}{2} = 0.95 \quad \text{for sediment-laden flows}
\end{aligned}$$



(b) The experimental velocity profiles and Equation (6.20) are plotted in the following figure. Note that Equation (6.13) is not compared because the flume experimental flows are smooth boundary flows.



Appendix MatLab Programs

```
%Program: Problem 6.6(a), PYJ Page 111
u_star = 0.041;   %m/s
d_s = 0.105;   %mm
Q = 0.064;  %m^3/s
h = 0.17;  %m
S_f = 0.002; W = 0.356;  %m

y = [6 12 18 24 30 46 69 91 122]; % 137 152 162];
u1 = [0.709 0.773 0.823 0.849 0.884 0.927 0.981 1.026 1.054];
u2 = [0.576 0.649 0.743 0.798 0.838 0.916 0.976 1.047 1.07];
y = y./122;
u1 = (1.054 - u1)./u_star;
```

```
u2 = (1.07 - u2)./u_star;

figure(1)
semilogx(y,u1,'o'), hold on
title('Clear Water')
sxlabel('y/{\delta}')
sylabel('\up{-5}\dfrac{u_{max}-u}{u_*}','Rot',0)

A = (u1(1)-u1(2))/log(y(1)/y(2));
C = u1(1) - A*log(y(1));
x = [y(1) 1];
y = A*log(x) + C;
semilogx(x,y)
line([0.117 0.25 0.25],[6.5 6.5 4.7])
stext(0.17,6.7,'1'), stext(0.25,5.5,'\dfrac{2.3}{\kappa}=5.2')

%Program: Problem 6.6(b), PYJ Page 111
figure(2)
y = y./1000; nu = 1e-6;
u = (2.5*log(u_star*y./nu) + 5.5)*u_star;
plot(u,y*1000), hold off
legend('Clear water data','Sediment-laden flow data','Equation (6.20)')
title('Comparison of velocity profiles')
xlabel('u (m/s)'), ylabel('z (mm)')
```

Problem 6.7

For the velocity profile given in Problem 6.1, calculate the depth-averaged velocity from (a) the velocity profile; (b) the one-point method; (c) the two point method; (d) the three-point method; and (e) the surface method.

Solution (a) The depth-average velocity is found that

$$\begin{aligned} V &= 0.9207 \text{ m/s} \quad \text{for clear water} \\ V &= 0.9014 \text{ m/s} \quad \text{for sediment-ladenflow} \end{aligned}$$

(b) The one-point velocity is

$$V = u_{0.4} = u|_{0.4\times122\text{mm}} = u|_{48.8\text{mm}} = 0.927 + \left(\frac{0.981 - 0.927}{69 - 46}\right)(48.8 - 46) = 0.934 \text{ m/s}$$

(c) The two-point velocity is

$$\begin{aligned} u|_{0.2\times122\text{mm}} &= u|_{24.4\text{mm}} = 0.849 \text{ m/s} \\ u|_{0.8\times122\text{mm}} &= u|_{97.6\text{mm}} = 1.026 + \left(\frac{1.054 - 1.026}{122 - 91}\right)(97.6 - 91) = 1.032 \text{ m/s} \end{aligned}$$

$$V = \frac{u|_{0.2\times 122\mathrm{mm}} + u|_{0.8\times 122\mathrm{mm}}}{2} = 0.940 \text{ m/s}$$

(d) The three-point velocity is

$$V = \frac{u|_{0.2\times 122\mathrm{mm}} + u|_{0.4\times 122\mathrm{mm}} + u|_{0.8\times 122\mathrm{mm}}}{3} = 0.938 \text{ m/s}$$

(e) The surface method gives that

$$V = 0.85 u_{\max} = 0.85\,(1.054) = 0.896 \text{ m/s}$$

The comparison is shown below.

	Measured	One-point	Two-point	Three-point	Surface
V (m)	0.927	0.934	0.940	0.938	0.896
Error (%)		0.75%	1.4%	1.2%	-3.3%

Appendix MatLab Program

```
%Program: Problem 6.7, PYJ Page 111
y = [6 12 18 24 30 46 69 91 122];
u1 = [0.709 0.773 0.823 0.849 0.884 0.927 0.981 1.026 1.054];
u2 = [0.576 0.649 0.743 0.798 0.838 0.916 0.976 1.047 1.07];
n = 9;

% clear water
S = u1(1)*y(2)/2;
for i = 2:n-1
 S = S +  u1(i)*(y(i+1)-y(i-1))/2;
end
S = S + u1(n)*(y(n)-y(n-1))/2;

V1 = S/122; %depth-averaged velocity

% sediment-lden flow
S = u2(1)*y(2)/2;
for i = 2:n-1
 S = S +  u2(i)*(y(i+1)-y(i-1))/2;
end
S = S + u2(n)*(y(n)-y(n-1))/2;

V2 = S/122; %depth-averaged velocity
```

Chapter 7

Incipient motion

Problem 7.1

What is the sediment size corresponding to beginning of motion when the shear velocity $u_* = 0.1$ m/s?

Solution The shear stress corresponding to $u_* = 0.1$ m/s is

$$\tau = \rho u_*^2 = (1000)\,(0.1)^2 = 10 \text{ N/m}^2$$

From Figure 7.7, we can get

$$d_s = 1.27 \text{ cm}$$

or assume that

$$\frac{\tau_c}{(\rho_s - \rho) g d_s} = 0.047$$

then we get

$$d_s = 1.31 \text{ cm}$$

Problem 7.2

Given the stream slope $S_0 = 10^{-3}$, at what flow depth would coarse gravel enter motion?

Solution For coarse gravel, we have $d_s = 16 - 32$ mm. Now

$$\frac{\tau_c}{(\rho_s - \rho) g d_s} = \frac{\rho g h S_0}{(\rho_s - \rho) g d_s} = \frac{h S_0}{(G-1)\, d_s} = 0.047$$

then

$$h = \frac{0.047\,(G-1)\,d_s}{S_0} = \frac{0.047\,(1.65)\,(16 \times 10^{-3})}{10^{-3}} = 1.24 \text{ m}$$

Problem 7.3

Calculate the stability factor of 8-in. riprap on an embankment inclined at a 1V: 2H sideslope if the shear stress $\tau_0 = 1$ lb/ft^2.

Solution

1. The particle size $d_s = 8$ in$= 0.667$ ft;
2. the angle of repose is approximately $\phi = 41.5°$ (from Fig. 7.2);
3. the sideslope is $\Theta_1 = \tan^{-1}\frac{1}{2} = 26.6°$;
4. the downstream slope is assumed to be $\Theta_0 = 0°$;
5. the angle $\Theta = \tan^{-1}\left(\frac{\sin\Theta_0}{\sin\Theta_1}\right) = 0$;
6. the factor $a_\Theta = \sqrt{\cos^2\Theta_1 - \sin^2\Theta_0} = \cos\Theta_1 = 0.895$;
7. the deviation angle $\lambda = 0$;
8. from Equation (7.11a), we have
$$\eta_0 = \frac{21\tau_0}{(\gamma_s - \gamma)\, d_s} = \frac{21\,(1)}{(1.65)\,(62.4)\,(0.667)} = 0.306$$
9. from Equation (7.13), assuming $M = N$, we have
$$\beta = \tan^{-1}\left\{\frac{\cos(\lambda+\Theta)}{\frac{2\sqrt{1-a_\Theta^2}}{\eta_0\tan\phi} + \sin(\lambda+\Theta)}\right\} = \tan^{-1}\left\{\frac{\cos 0°}{\frac{2\sqrt{1-0.895^2}}{(0.306)\tan 41.5°} + \sin 0°}\right\} = 16.88°$$
10. from Equation (7.10), we have
$$\eta_1 = \eta_0\left[\frac{1+\sin(\lambda+\beta+\Theta)}{2}\right] = 0.306\left[\frac{1+\sin 16.18°}{2}\right] = 0.197$$
11. from (7.8), because $\lambda \geq 0$,
$$\begin{aligned}\mathrm{SF}_0 &= \frac{a_\Theta\tan\phi}{\eta_1\tan\phi + \sqrt{1-a_\Theta^2}\cos\beta} = \frac{0.895\tan 41.5°}{(0.197)\tan 41.5° + \sqrt{1-0.895^2}\cos 16.88°}\\ &= \frac{0.792}{0.174+0.427} = 1.32 > 1\end{aligned}$$
then the particle is stable.

Problem 7.4

An angular 10-mm sediment particle is submerged on an embankment inclined at $\Theta_1 = 20°$ and $\Theta_0 = 0°$. Calculate the critical shear stress from the moment stability method when the streamlines near the particle are (a) $\lambda = 15°$ (deflected downward); (b) $\lambda = 0°$ (horizontal flow); and (c) $\lambda = -15°$ (deflected upward).

Solution This problem is to simultaneously solve Eqs. (7.8b), (7.10) and (7.13). The MatLab Program and results are shown in Problem 7.5.

Problem 7.5

Compare the values of critical shear stress $\tau_{\Theta c}$ from Problem 7.4 with those calculated with Equation (7.16) and with Lane's method [Eq. (7.18)], given $\Theta_0 = 0$ and $\Pi_{1d} = 0$.

Solution The values are given in the following tabulation:

Angle λ (deg)	Moment stability (N/m^2)	Simplified stability (N/m^2)	Lane's method (N/m^2)
-15° up	6.58	7.22	6.26
0	5.69	6.26	6.26
15° down	5.00	5.43	6.26

MATLAB Program

```
% Problem 7.5: Find critical shear stress
% Method in Section 7.3

lambda = 15; % degree
Pi_ld = 0;
d_s = 10e-3; % m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Section 7.3

phi = 37; % degree
theta_1 = 20; % degree
theta_0 = 0; % degree

x = fsolve('p7_5f',[1 1 1]',1,[],lambda,d_s,phi,theta_0,theta_1);

beta = x(1);
eta_0 = x(2);
eta_1 = x(3);
```

```
method = 'Moment method'
tau_0 = eta_0*(2650-1000)*9.81*d_s*0.047

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Simplified method (Section 7.4)

phi = radian(phi); % radian
theta_0 = radian(theta_0); % radian
theta_1 = radian(theta_1);  % radian
lambda = radian(lambda); % radian

a = sin(theta_1)*sin(lambda)+sin(theta_0)*cos(lambda);
b =tan(phi)*sqrt(1+Pi_ld.^2);
c = sin(theta_0).^2 + sin(theta_1).^2;
d = sin(phi).^2;

ratio = -a/b + sqrt((a/b).^2 +1 - c/d);

method = 'Simplified method'
tau_tc = ratio*0.047*(2650-1000)*9.81*d_s

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Lane's method Eq.(7.18), p.127

method = 'Lane''s method'
tau_tc = sqrt(1-(sin(theta_1)/sin(phi)).^2)*0.047*(2650-1000)*9.81*d_s

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%% FUNCTION %%%%%%%%%%%%%%%%%%%%%
function f = p7_5f(x,lambda,d_s,phi,theta_0,theta_1)

beta = x(1); eta_0 = x(2); eta_1 = x(3);

lambda = radian(lambda); % radian
phi = radian(phi); % radian
theta_0 = radian(theta_0); % radian
theta_1 = radian(theta_1); % radian

theta = atan(sin(theta_0)/sin(theta_1));
a_theta = sqrt(cos(theta_1).^2 - sin(theta_0).^2);

f(1) = tan(beta) - cos(lambda + theta)./(2*sqrt(1-...
       a_theta.^2)./eta_0*tan(phi) + sin(lambda + theta));
```

```
f(2) = eta_1./eta_0 - 0.5*(1 + sin(lambda + beta + theta));
f(3) = a_theta*tan(phi) - (eta_1*tan(phi) - sqrt(1 -...
       a_theta.^2)*cos(beta));
```

Problem 7.6

Design a stable channel conveying 14 m^3/s in coarse gravel, $d_{50} = 10$ mm and $d_{90} = 20$ mm, at a slope $S_0 = 0.0006$.

Solution (1) Find the flow depth. The flow depth is set such that the bed particles are at incipient motion, assuming $R_h = h$.

$$\tau_b = 0.97\rho g h S_0 = 0.06(\rho_s - \rho) g d_{50} \tan\phi$$

in which $\phi = 36.5°$ from Fig. 7.1.

$$h = \frac{0.06(G-1)d_{50}\tan\phi}{0.97S_0} = \frac{0.06\,(1.65)\,(0.01)\,(\tan 36.5°)}{0.97\,(0.0006)} = 1.26 \text{ m}$$

(2) Determine the sideslope angle Θ_1. Using Lane's method, we have

$$\tau_s = 0.75\rho g h S_0 = 0.06(\rho_s - \rho) g d_{50} \tan\phi \sqrt{1 - \left(\frac{\sin\Theta_1}{\sin\phi}\right)^2}$$

which gives that

$$\begin{aligned}
\sin\Theta_1 &= \sin\phi \sqrt{1 - \left(\frac{0.75hS_0}{0.06(G-1)d_{50}\tan\phi}\right)^2} \\
&= \sin 36.5° \sqrt{1 - \left(\frac{0.75\,(1.26)\,(0.0006)}{0.06\,(G-1)\,(0.01)\tan 36.5°}\right)^2} \\
&= 0.377
\end{aligned}$$

or

$$\Theta_1 = 22.1°$$

(3) Determine the channel width. The cross-sectional area of a trapezoidal channel is

$$A = Bh + h^2 \cot\Theta_1$$

in which B is width of the bottom. The perimeter is

$$P = B + 2h/\sin\Theta_1$$

The hydraulic radius is

$$R_h = \frac{A}{P} = \frac{Bh + h^2\cot\Theta_1}{B + 2h/\sin\Theta_1} \tag{7.1}$$

Then the velocity is

$$V = \frac{Q}{A} = \frac{Q}{h\,(B + h\cot\Theta_1)} \tag{7.2}$$

From Eq. (6.14), we get

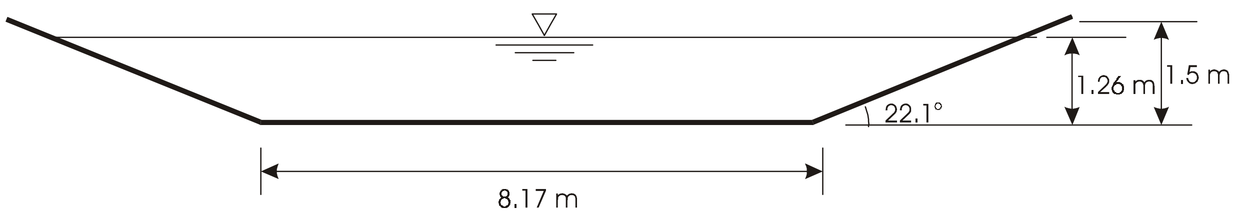
$$V = \sqrt{gR_hS_0}\left[2.5\ln\left(\frac{12.2R_h}{3d_{90}}\right)\right] \tag{7.3}$$

From (7.2) and (7.3), we get

$$\frac{Q}{h\,(B + h\cot\Theta_1)} = \sqrt{gR_hS_0}\left[2.5\ln\left(\frac{12.2R_h}{3d_{90}}\right)\right] \tag{7.4}$$

Now solving (7.1) and (7.4) simultaneously for R_h and B given $Q = 14\ \mathrm{m^3/s}$, $d_{90} = 0.02$ m and $S = 0.0006$ requires

$$R_h = 0.95\ \mathrm{m}, \quad B = 8.17\ \mathrm{m}$$



Problem 7.6: Stable trapezoidal channel

MATLAB Program

```
% Problem 7.6: Stable channel design

Q = 14;  % m^3/s
d_90 = 0.02; % m
d_50 = 0.01; % m
phi = radian(36.5); % degree
S_0 = 0.0006;

% Find the flow depth
h = 0.06*1.65*d_50*tan(phi)/(0.97*S_0);

% Find the sideslope angle
theta_1 = asin(sin(phi)*sqrt(1 - ...
          (0.75*S_0*h/0.06/1.65/d_50/tan(phi)).^2));
```

```
% Determine the channel bottom
x = fsolve('p7_6f',[1 10]',1,[],h,theta_1,Q,S_0,d_90);

h
theta_1 = 180/pi*theta_1
R_h = x(1);
B = x(2)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function f = p7_6f(x,h,theta_1,Q,S_0,d_90)

R_h = x(1);
B = x(2);

f(1) = B*h + h.^2*cot(theta_1) - B*R_h - 2*h*R_h/sin(theta_1);
f(2) = Q/h/(B + h*cot(theta_1)) ...
       - sqrt(9.81*R_h*S_0)*2.5*log(12.2*R_h/3/d_90);
```

Chapter 8

Bedforms

Exercise

1. Demonstrate that Equations (8.1) and (8.2) reduce to Equations (8.3) and (8.4).

   Solution

$$\frac{\partial}{\partial x}\left(\frac{p}{\rho_m}+\frac{v^2}{2}\right)=g_x+(v_y\otimes_z-v_z\otimes_y)+\frac{1}{\rho_m}\left(\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}\right) \quad (8.1)$$

$$\frac{\partial}{\partial z}\left(\frac{p}{\rho_m}+\frac{v^2}{2}\right)=g_z+(v_x\otimes_y-v_y\otimes_x)+\frac{1}{\rho_m}\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}\right) \quad (8.2)$$

Assumptions:

$$\tau_{xx}=\tau_{yx}=\tau_{yz}=\tau_{zz}=0$$

$$v_x=v,\quad v_y=v_z=0$$

Substituting the above relations into (8.1) gives that

$$g\frac{\partial}{\partial x}\left(\frac{p}{\rho_m g}+\frac{v^2}{2g}\right)=g_x+\frac{1}{\rho_m}\frac{\partial\tau_{zx}}{\partial z} \quad (8.3)$$

Substituting the above relations into (8.2) and considering

$$\otimes_y=\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}=\frac{\partial v_x}{\partial z}$$

gives that

$$\frac{\partial}{\partial z}\left(\frac{p}{\rho_m}+\frac{v^2}{2}\right)=g_z+v_x\frac{\partial v_x}{\partial z}+\frac{1}{\rho_m}\frac{\partial\tau_{xz}}{\partial x} \quad (8.4)$$

Problem 8.1

Identify the conditions under which the pressure and shear stress distributions vary linearly with flow depth.

Answer. Steady uniform flow.

Problem 8.2

Determine the flow regime and type of bedform in the Rio Grande conveyance channel, given the mean velocity $V = 0.5$ m/s, the flow depth $h = 0.40$ m, the bed slope $S_0 = 52$ cm/km, and the grain size distributions $d_{50} = 0.24$ mm and $d_{65} = 0.35$ mm.

Solution The stream power is

$$\tau_0 V = \gamma h S_0 V = (62.4)(0.4/0.3048)(0.52/1000)(0.5/0.3048) = 0.07 \text{ lb/ft-s}$$

According to Simons and Richardson's method (Figure 8.8), the bedforms are ripples.

Problem 8.3

Check the type of bedform in a 200-ft-wide channel conveying 8500 ft^3/s in a channel sloping at 9.6×10^{-5} given the mean velocity $V = 3.6$ ft/s and the median grain diameter $d_m = 0.213$ mm.

Solution The flow depth is

$$h = \frac{Q}{WV} = \frac{8500}{(200)(3.6)} = 11.8 \text{ ft}$$

the stream power is

$$\tau_0 V = \gamma h S_0 V = (62.4)(11.8)\left(9.6 \times 10^{-5}\right)(3.6) = 0.254 \text{ lb/ft-s}$$

According to Simons and Richardson's method (Figure 8.8), the bedforms are dunes.

Problem 8.4

Predict the type and geometry of bedforms in a sand-bed channel, $d_{35} = 0.35$ mm. and $d_{65} = 0.42$ mm, sloping at $S_0 = 0.001$ with flow depth $h = 1$ m when the water temperature is 40°F.

Solution According to the method of Engelund and Hanson (Figure 8.12), we have

$$\tau_* = \frac{\tau_0}{(\gamma_s - \gamma)\, d_{65}} = \frac{\gamma h S_0}{(\gamma_s - \gamma)\, d_{65}} = \frac{h S_0}{(G-1)\, d_{65}} = \frac{(1)(0.001)}{(1.65)(0.42 \times 10^{-3})} = 1.44$$

Therefore, the bedforms are in the transition from plane bed to antidunes.

Problem 8.5

A 20-m-wide alluvial channel conveys a discharge $Q = 45$ m^3/s. If the channel slope is $S_0 = 0.0003$ and the median sediment size is $d_m = 0.4$ mm, determine (a) the flow depth from Engelund's method; (b) the type of bedform; and (c) the bedform geometry from van Rijn's method.

Solution (a) **Engelund Method**

The dune data in Figure 8.12 can be approximated with

$$\frac{\gamma h' S_0}{(\gamma_s - \gamma)d_s} = 0.06 + 0.4\left(\frac{\gamma h S_0}{(\gamma_s - \gamma)d_s}\right) \tag{p8.5a}$$

From Eq. (6.14), we have the skin friction

$$\frac{V}{\sqrt{gh'S_0}} = 2.5\ln\left(\frac{h'}{2d_m}\right) + 6 \tag{p8.5b}$$

The continuity equation is

$$\text{Continuity}: \qquad Q = VhB \tag{p8.5c}$$

in which $\tau'_* = \dfrac{\gamma h' S_0}{(\gamma_s - \gamma)d_s}$, and $\tau_* = \dfrac{\gamma h S_0}{(\gamma_s - \gamma)d_s}$ while h, h' and V are unknowns.

Solution Procedures: (1) Assume h, find h' from (p8.5a); (2) Calculate V from (p8.5b); (3) Calculate h from (p8.5c); and (4) Compare h assumed with calculated. If they are not close, repeat the calculations. A MATLAB has been written to carry out the calculation.

Given: $Q = 45$ m^3/s, $B = 20$ m, $S_0 = 3\times10^{-4}$, and $d_m = 0.4$ mm. The calculated results are

$h = 1.954$ m	$h' = 0.826$ m	$V = 1.15$ m

MATLAB PROGRAM FOR ENGELUND METHOD

function table = engelund(Q,B,ds,Se)

```
% [h,h1,V] = engelund(Q,B,ds,S): Calculate flow depth
%             using Engelund method
% h = total hydraulic radius;
% h1 = sand hydraulic radius;
% V = mean velocity in m/s;
% Q = discharge in m^3/s;
% B = channel width in m;
% ds = sediment size in m;
% Se = energy slope.
%
% Written by JUNKE GUO, Nov. 12, 1996
%
h = 1;  %assumed flow depth
hh = 3;  %another assumed flow depth. MATLAB while.
G = 2.65;
```

```
tau1 = 0.05;
Fr = 0.2;
while abs(hh-h) > 1e-4
      h = (hh+h)./2;
      tau = h.*Se./(G-1)./ds;

      if Fr < 1
       if tau1 < 0.05
         tau1 = tau;
       else
         tau1 = 0.06+0.4.*tau.^2;
       end
      end

      if Fr >= 1
        if tau1 >= 0.55 & tau1 < 1
         tau1 = tau;
        else
         tau1 = (0.702.*tau.^(-1.8)+0.298).^(-1./1.8);
        end
      end

      h1 = tau1.*(G-1).*ds./Se;
      V = sqrt(9.81.*h1.*Se).*2.5.*log(5.5.*h1./ds);
      hh = Q./V./B;
      Fr = V./sqrt(9.81.*hh);
end

table = [h h1 V];
```

(b) **Bedform Prediction**

$$d_* = d_s \left[\frac{(G-1)\, g}{\nu^2} \right]^{1/3}; \qquad T \text{ from Eq. (8.9b)}$$

$$\boxed{d_* = 10.1 \qquad T = 9.43 \quad \text{Bedform} \;=\; \text{Dune}}$$

MATLAB PROGRAM

```
function s = bedform(ds,Tem,h1,S)
%bedform(ds,Tem,h1,S): Bedform classification by van Rijn's method
%
% ds = sediment diameter in m;
% Tem = temparature;
```

```
% h1 = sand hydraulic radius;
% S = slope;
% T = transport-stage function;
% output = Ripple or Dune or Transition or Upper regime
%
% Written by JUNKE GUO, Nov.12, 1996
% Reference: Julien, Erosion and sedimentation, p145

G = 2.65; g = 9.81;
nu = (1.14-0.031.*(Tem-15)+0.00068.*(Tem-15).^2).*1e-6;
dstar = ds.*((G-1).*g/nu.^2).^(1./3)

if dstar < 0.3
   taustarc = 0.5.*tan(pi./6);
elseif dstar >= 0.3 & dstar < 19
   taustarc = 0.25.*dstar.^(-0.6).*tan(pi./6);
elseif dstar >=19 & dstar < 50
   taustarc = 0.06.*tan(pi./6);
end

taustar1 = h1.*S./(G-1)./ds;
T = taustar1./taustarc - 1;
if dstar < 10 & T < 3, s = 'Ripple'; end
if dstar >= 10 | T >= 3 & T < 15, s = 'Dune'; end
if T >= 15 & T < 25, s = 'Transition'; end
if T >= 25, s = 'Upper regime'; end
```

(c) **Bedform Geometry**

$$\Lambda = 7.3h = (7.3)\,(1.95) = 14.2 \text{ m}$$

$$\frac{\Delta}{h} = 0.11\left(\frac{d_{50}}{h}\right)^{0.3}\left(1 - e^{-0.5T}\right)(25 - T)$$

$$\Delta = (1.95)\,(0.11)\left(\frac{4\times10^{-4}}{1.95}\right)^{0.3}\left(1 - e^{-0.5(9.43)}\right)(25 - 9.43) = 0.259 \text{ m}$$

i.e.

$$\boxed{\Lambda = 14.2 \text{ m} \qquad \Delta = 0.259 \text{ m}}$$

Problem 8.6

Dunes 6.4 m in height and 518 m in length were measured in the Mississippi River. If the flow depth is 38.7 m, the river energy slope 7.5 cm/km, the water temperature

16°C, the depth-averaged flow velocity 2.6 m/s, and the grain size $d_{50} = 0.25$ mm and $d_{90} = 0.59$ mm: (a) check the type of bedform predicted by the methods of Liu, Chabert and Chauvin, Simons and Richardson, Bogardi, and van Rijn; (b) estimate τ_b'' from Equation (8.13); and (c) calculate k_s, from field data using Equation (8.15) and plot the results in Figure CS8.1.3.

Solution (a) Given

$$\begin{aligned}\Delta &= 6.4 \text{ m}, \quad \Lambda = 518 \text{ m}, \quad h = 38.7 \text{ m}, \quad S_f = 7.5 \times 10^{-5}, \quad T = 16°\text{C} \\ V &= 2.6 \text{ m/s}, \quad d_{50} = 0.25 \text{ mm}, \quad d_{90} = 0.59 \text{ mm}\end{aligned}$$

we have the following parameters:

$$\begin{aligned}\nu &= 1.16 \times 10^{-6} \text{ m}^2/\text{s} \\ \omega &= 33.65 \text{ mm/s}\end{aligned}$$

$$u_* = \sqrt{ghS_f} = \sqrt{(9.81)(38.7)(7.5 \times 10^{-5})} = 0.169 \text{ m/s}$$

$$d_* = d_{50}\left[\frac{(G-1)g}{\nu^2}\right]^{1/3} = \left(0.25 \times 10^{-3}\right)\left[\frac{(1.65)(9.81)}{(1.16 \times 10^{-6})^2}\right]^{1/3} = 5.73$$

$$\text{Re}_* = \frac{u_* d_{50}}{\nu} = \frac{(0.169)(0.25 \times 10^{-3})}{1.16 \times 10^{-6}} = 36.4$$

$$\frac{u_*}{\omega} = \frac{0.169}{0.03365} = 5.07$$

$$\tau_* = \frac{\tau_0}{(\gamma_s - \gamma) d_{50}} = \frac{\gamma h S_f}{(\gamma_s - \gamma) d_{50}} = \frac{h S_f}{(G-1) d_{50}} = \frac{(38.7)(7.5 \times 10^{-5})}{(1.65)(0.25 \times 10^{-3})} = 7.03$$

$$\frac{g d_{50}}{u_*^2} = \frac{0.6}{\tau_*} = \frac{0.6}{7.03} = 0.085$$

$$\tau_0 V = \gamma h S_f V = (62.4)(38.7/0.3048)\left(7.5 \times 10^{-5}\right)(2.6/0.3048) = 5.07 \text{ lb/ft-s}$$

$$\begin{aligned}T &= \frac{\tau_*' - \tau_{*c}}{\tau_{*c}} = \frac{u_*'^2 - u_{*c}^2}{u_{*c}^2} = \frac{\rho V^2}{\tau_c\left[5.75 \log\left(4h/d_{90}\right)\right]^2} - 1 \\ &= \frac{(1000)(2.6)^2}{0.047(1650)(9.81)(0.25 \times 10^{-3})\left[5.75 \log\left(\frac{4(38.7)}{0.59 \times 10^{-3}}\right)\right]^2} - 1 \\ &= 35.6\end{aligned}$$

Liu's method: From Figure 8.6, given $\text{Re}_* = 36.4$ and $u_*/\omega = 5.07$, the bedforms are antidunes. **Chabert and Chauvin**'s method: From Figure 8.7, given $\tau_* = 6.87$ and $\text{Re}_* = 36.4$, the bedforms are off the chart. **Simons and Richardson**'s method: Given $\tau_0 V = 5.07$ lb/ft-s and $d_s = 0.25$ mm, from Figure 8.8, we get antidunes. **Bogardi**'s method: Given $g d_{50}/u_*^2 = 0.0873$ and $d_s = 0.25$ mm, from Figure 8.9, we get antidunes. **van Rijn**'s method: Given $T = 35.6$ and $d_* = 5.73$, from Figure 8.10, we get antidunes. **Discussion**: All methods give antidunes, except perhaps Chabert and Chauvin. This shows that most existing bedform predictors cannot be applied to large rivers.

(b) From Figure 8.12, given $\tau_* = 6.87$ (antidunes), we have $\tau'_* = 1.91$. Then

$$\tau''_* = \tau_* - \tau'_* = 6.87 - 1.91 = 4.96$$

$$\tau'' = (\gamma_s - \gamma)\, d_{50}\tau''_* = (1650)\,(9.81)\left(0.25 \times 10^{-3}\right)(4.96) = 20 \text{ N/m}^2$$

(c)

$$\sqrt{\frac{8}{f}} = \frac{V}{u_*} = \frac{2.6}{0.169}, \quad f = 0.0338$$

From Equation (8.15), we have

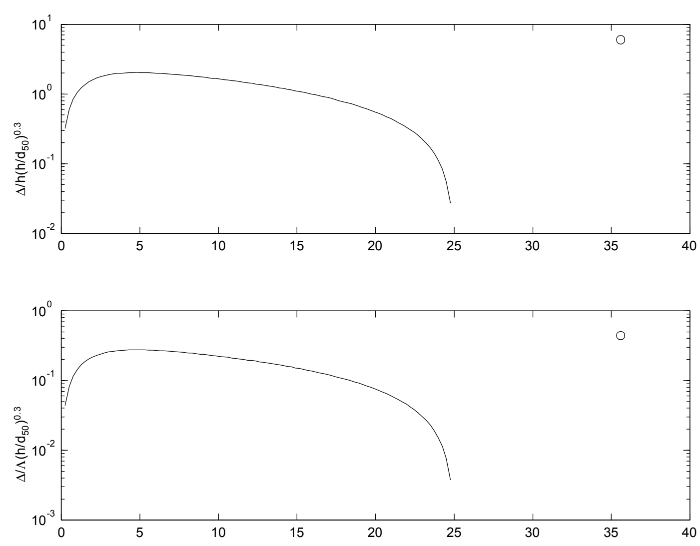
$$\sqrt{\frac{8}{0.0338}} = 5.75 \log \frac{12\,(38.7)}{k_s}, \quad k_s = 0.98 \text{ m}$$

Since we have $\Lambda = 518$ m and $\Delta = 6.4$, then

$$\frac{\Delta}{h}\left(\frac{d_{50}}{h}\right)^{-0.3} = \frac{6.4}{38.7}\left(\frac{0.00025}{38.7}\right)^{-0.3} = 5.96$$

$$\frac{\Delta}{\Lambda}\left(\frac{d_{50}}{h}\right)^{-0.3} = \frac{6.4}{518}\left(\frac{0.00025}{38.7}\right)^{-0.3} = 0.44$$

which is plotted in Figure CS8.1.3 as shown below.



MatLab Program

```
% Problem 8.6
T = 35.6; y1 = 5.96; y2 = 0.44;
figure(1), subplot(211), semilogy(T,y1,'o'), hold on
x = linspace(0,25,100);
y1 = 0.11.*(1-exp(-0.5.*x)).*(25-x);
semilogy(x,y1), hold off
sylabel('\Delta/h(h/d_{50})^{0.3}')
subplot(212), semilogy(T,y2,'o'), hold on

y2 = 0.015.*(1-exp(-0.5.*x)).*(25-x);
semilogy(x,y2), hold off
sxlabel('Transport stage parameter T')
sylabel('\Delta/\Lambda(h/d_{50})^{0.3}')
subplot
```

Problem 8.7

Surveys of the Red Deer River indicate the formation of sand dunes 0.6 m high and 5.2 m long in bed material $d_{50} = 0.34$ mm and $d_{90} = 1.2$ mm. If the flow depth is 2 m, the average flow velocity 0.95 m/s, and the slope 7.4 cm/km, (a) compare the bedform characteristics with those of Liu, Chabert and Chauvin, Simons and Richardson, Bogardi, Yalin, Engelund, and van Rijn; (b) separate the total bed shear stress τ_b into grain and form shear stress components τ_b' and τ_b'' using the methods of Engelund and van Rijn; and (c) compare the bedform length and height with the methods of Yalin and van Rijn.

Solution (a) Measurement data:

$$\begin{aligned}\Delta &= 0.6 \text{ m}, \quad \Lambda = 5.2 \text{ m}, \quad d_{50} = 0.34 \text{ mm}, \quad d_{90} = 1.2 \text{ mm} \\ h &= 2 \text{ m}, \quad V = 0.95 \text{ m/s}, \quad S_0 = 7.4 \times 10^{-5}\end{aligned}$$

Assume that the temperature is $T = 20°$C, the kinematic viscosity is

$$\nu = 1.01 \times 10^{-6} \text{ m}^2/\text{s}$$

and the fall velocity is

$$\omega = 48.35 \text{ mm/s}$$

The following parameters are calculated:

$$u_* = \sqrt{ghS_f} = \sqrt{(9.81)\,(2)\,(7.4 \times 10^{-5})} = 0.0381 \text{ m/s}$$

$$d_* = d_{50}\left[\frac{(G-1)\,g}{\nu^2}\right]^{1/3} = \left(0.34 \times 10^{-3}\right)\left[\frac{(1.65)\,(9.81)}{(1.01 \times 10^{-6})^2}\right]^{1/3} = 12.8$$

$$\mathrm{Re}_* = \frac{u_* d_{50}}{\nu} = \frac{(0.0381)\,(0.34 \times 10^{-3})}{1.01 \times 10^{-6}} = 56.7$$

$$\frac{u_*}{\omega} = \frac{0.0381}{0.04835} = 0.788$$

$$\tau_* = \frac{\tau_0}{(\gamma_s - \gamma)\, d_{50}} = \frac{\gamma h S_f}{(\gamma_s - \gamma)\, d_{50}} = \frac{h S_f}{(G-1)\, d_{50}} = \frac{(2)\left(7.4 \times 10^{-5}\right)}{(1.65)\left(0.34 \times 10^{-3}\right)} = 0.264$$

$$\frac{g d_{50}}{u_*^2} = \frac{0.6}{\tau_*} = \frac{0.6}{0.264} = 2.27$$

$$\tau_0 V = \gamma h S_f V = (62.4)\,(2/0.3048)\left(7.4 \times 10^{-5}\right)(0.95/0.3048) = 0.0944 \text{ lb/ft-s}$$

$$\begin{aligned} T &= \frac{\tau_*' - \tau_{*c}}{\tau_{*c}} = \frac{u_*'^2 - u_{*c}^2}{u_{*c}^2} = \frac{\rho V^2}{\tau_c \left[5.75 \log\left(4h/d_{90}\right)\right]^2} - 1 \\ &= \frac{(1000)\,(0.95)^2}{0.047\,(1650)\,(9.81)\left(0.34 \times 10^{-3}\right)\left[5.75 \log\left(\dfrac{4\,(2)}{1.2 \times 10^{-3}}\right)\right]^2} - 1 \\ &= 6.22 \end{aligned}$$

Liu's method: From Figure 8.6, given $\mathrm{Re}_* = 12.8$ and $u_*/\omega = 0.788$, the bedforms are dunes. **Chabert and Chauvin**'s method: From Figure 8.7, given $\tau_* = 0.264$ and $\mathrm{Re}_* = 12.8$, the bedforms are ripples to dunes. **Simons and Richardson**'s method: Given $\tau_0 V = 0.0944$ lb/ft-s and $d_s = 0.34$ mm, from Figure 8.8, we get dunes. **Bogardi**'s method: Given $g d_{50}/u_*^2 = 2.27$ and $d_s = 0.34$ mm, from Figure 8.9, we get dunes. **Engelund**'s method: Given $\tau_* = 0.264$, the bedforms are dunes. **van Rijn**'s method: Given $T = 6.22$ and $d_* = 8.54$, from Figure 8.10, we get dunes. **Discussion**: All methods give dune bedforms for this river. This shows that existing bedform predict methods can be applied to small rivers.

(b) Engelund's method: From Figure 8.12, given $\tau_* = 0.264$ (dunes), we have $\tau_*' = 0.0879$. Then

$$\tau_*'' = \tau_* - \tau_*' = 0.264 - 0.0879 = 0.176$$

$$\tau'' = (\gamma_s - \gamma)\, d_{50} \tau_*'' = (1650)\,(9.81)\left(0.34 \times 10^{-3}\right)(0.176) = 0.969 \text{ N/m}^2$$

$$\tau' = \tau - \tau'' = \gamma h S_0 - \tau'' = (9810)\,(2)\left(7.4 \times 10^{-5}\right) - 0.969 = 0.483 \text{ N/m}^2$$

(c) Yalin's method: The dune length is

$$\Lambda = 2\pi h = 2\pi\,(2) = 12.6 \text{ m}$$

van Rijn's method: The dune height is

$$\begin{aligned} \Delta &= 0.11 h \left(\frac{d_{50}}{h}\right)^{0.3} \left(1 - e^{-0.5T}\right)(25 - T) \\ &= 0.11\,(2)\left(\frac{1.2 \times 10^{-3}}{2}\right)^{0.3} \left(1 - e^{-0.5(6.22)}\right)(25 - 6.22) \\ &= 0.426 \text{ m} \end{aligned}$$

The dune length is

$$\Lambda = 7.3h = 7.3\,(2) = 14.6 \text{ m}$$

Computer Problem 8.1

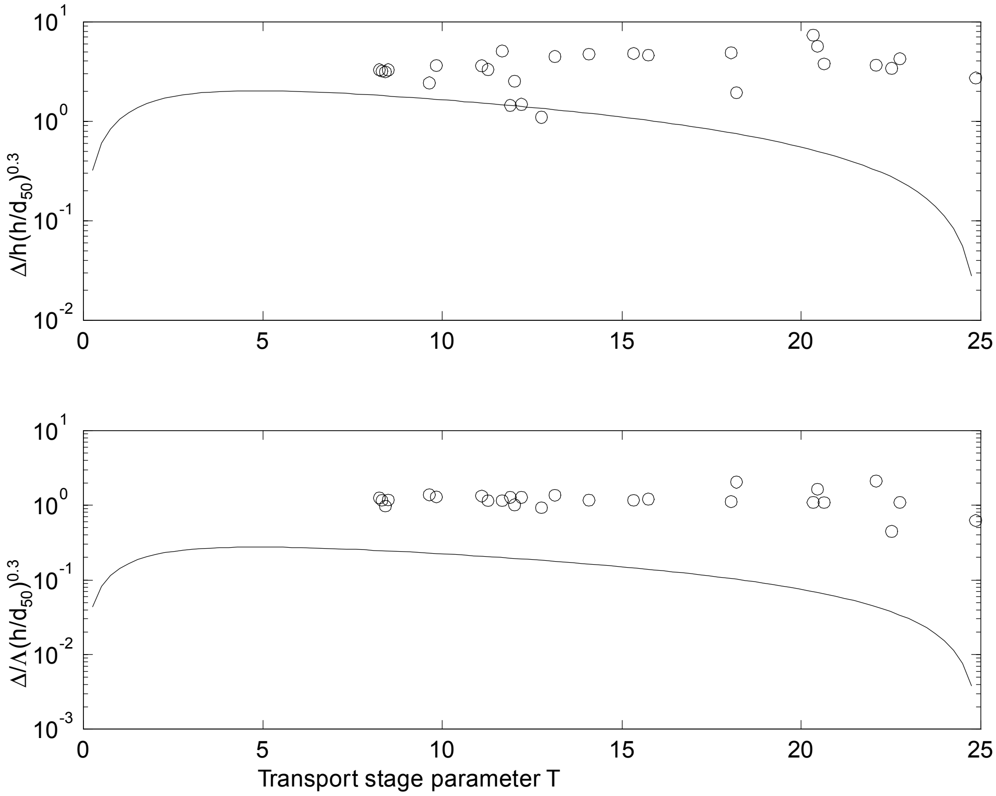
From the Bergsche Maas bedform data given in the tabulation on the following page, (a) calculate the grain Chezy coefficient C' and the sediment transport parameter T; (b) plot the data on van Rijn's dune height and dune steepness diagrams; and (c) calculate k_s, from field measurements and plot in Figure 8.13.

Q (m^3/s)	S_f (cm/km)	h (m)	V (m/s)	d_s (μm)	Δ (m)	Λ (m)	T	C' ($m^{1/2}/s$)
2160.00	12.50	8.60	1.35	480	1.50	22.50	**8.2**	**86.8**
2160.00	12.50	8.00	1.35	410	1.00	14.00	**9.6**	**87.6**
2160.00	12.50	10.50	1.30	300	1.50	30.00	**11.3**	**91.8**
2160.00	12.50	10.00	1.40	500	1.60	32.00	**8.4**	**87.4**
2160.00	12.50	7.60	1.40	520	1.40	21.00	**8.5**	**85.5**
2160.00	12.50	8.40	1.40	380	1.50	22.50	**11.1**	**88.5**
2160.00	12.50	8.70	1.70	300	1.50	30.00	**20.6**	**90.5**
2160.00	12.50	7.50	1.55	250	2.50	50.00	**20.3**	**91.0**
2160.00	12.50	8.30	1.50	260	1.80	36.00	**18.0**	**91.3**
2160.00	12.50	9.50	1.35	230	1.80	36.00	**15.7**	**93.2**
2160.00	12.50	8.80	1.35	240	1.80	36.00	**15.3**	**92.4**
2160.00	12.50	9.00	1.30	240	1.80	36.00	**14.1**	**92.6**
2160.00	12.50	9.60	1.50	220	2.20	33.00	**20.5**	**93.6**
2160.00	12.50	8.70	1.50	370	1.90	28.50	**13.1**	**88.9**
2160.00	12.50	8.20	1.35	330	2.00	36.00	**11.7**	**89.5**
2160.00	12.50	8.20	1.35	480	1.40	22.40	**8.3**	**86.5**
2160.00	12.50	8.10	1.40	350	1.00	20.00	**12.0**	**88.9**
2160.00	12.50	8.00	1.50	420	0.60	9.00	**11.9**	**87.4**
2160.00	12.50	7.80	1.50	410	0.60	9.00	**12.2**	**87.4**
2160.00	12.50	6.80	1.50	400	0.40	8.00	**12.8**	**86.6**
2160.00	12.50	6.40	1.50	270	0.60	6.00	**18.2**	**89.3**
2160.00	12.50	5.80	1.50	220	1.00	10.00	**22.1**	**90.2**
2160.00	12.50	6.20	1.50	210	1.20	24.00	**22.7**	**91.0**
2160.00	12.50	6.60	1.50	210	1.00	50.00	**22.5**	**91.5**
2160.00	12.50	8.30	1.50	180	0.90	36.00	**24.9**	**94.2**
2160.00	12.50	8.10	1.35	400	1.50	22.50	**9.8**	**87.9**

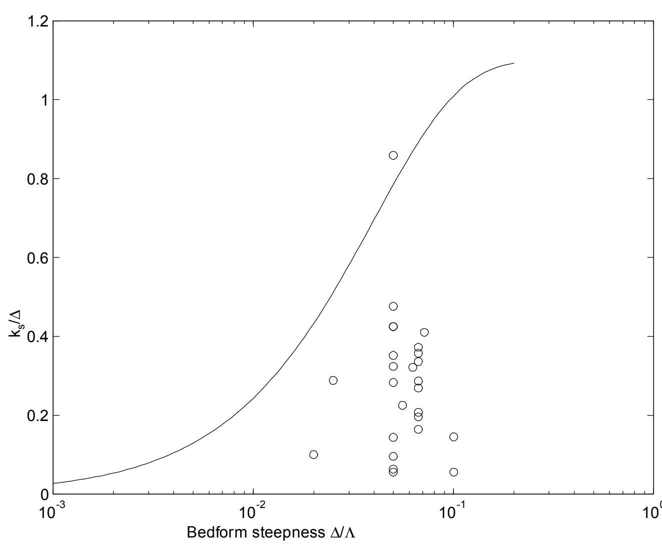
Source: After Julien (1992)

Solution (a) The values of the grain Chezy coefficient C' and the sediment transport parameter T are shown in the last two columns of the above table.

(b) The plots of the dune height and length are shown in the following figures.



(c) The plot of the field data is shown in the following figure.



MATLAB Program

```
A = [2160.00 12.50    8.60   1.35 480 1.50 22.50
```

```
    2160.00 12.50    8.00   1.35 410 1.00 14.00
    2160.00 12.50   10.50   1.30 300 1.50 30.00
    2160.00 12.50   10.00   1.40 500 1.60 32.00
    2160.00 12.50    7.60   1.40 520 1.40 21.00
    2160.00 12.50    8.40   1.40 380 1.50 22.50
    2160.00 12.50    8.70   1.70 300 1.50 30.00
    2160.00 12.50    7.50   1.55 250 2.50 50.00
    2160.00 12.50    8.30   1.50 260 1.80 36.00
    2160.00 12.50    9.50   1.35 230 1.80 36.00
    2160.00 12.50    8.80   1.35 240 1.80 36.00
    2160.00 12.50    9.00   1.30 240 1.80 36.00
    2160.00 12.50    9.60   1.50 220 2.20 33.00
    2160.00 12.50    8.70   1.50 370 1.90 28.50
    2160.00 12.50    8.20   1.35 330 2.00 36.00
    2160.00 12.50    8.20   1.35 480 1.40 22.40
    2160.00 12.50    8.10   1.40 350 1.00 20.00
    2160.00 12.50    8.00   1.50 420 0.60  9.00
    2160.00 12.50    7.80   1.50 410 0.60  9.00
    2160.00 12.50    6.80   1.50 400 0.40  8.00
    2160.00 12.50    6.40   1.50 270 0.60  6.00
    2160.00 12.50    5.80   1.50 220 1.00 10.00
    2160.00 12.50    6.20   1.50 210 1.20 24.00
    2160.00 12.50    6.60   1.50 210 1.00 50.00
    2160.00 12.50    8.30   1.50 180 0.90 36.00
    2160.00 12.50    8.10   1.35 400 1.50 22.50];

h = A(:,3); V = A(:,4); d_90 = 1e-6*A(:,5); Delta = A(:,6); Lambda = A(:,7);

% (a)
Q = 2160;  B = Q./h./V;  R_h = h.*B./(2*h+B);
C1 = 2.5*sqrt(9.81).*log(4.*R_h./d_90);
T = 1000.*V.^2./0.033/1650/9.81./d_90./(C1./sqrt(9.81)).^2 - 1;

% (b)
y1 = Delta./h.*(d_90./h).^(-0.3);
y2 = Delta./Lambda.*(d_90./h).^(-0.3);
figure(1)
subplot(211), semilogy(T,y1,'o'), hold on
x = linspace(0,25,100);
y1 = 0.11.*(1-exp(-0.5.*x)).*(25-x);
semilogy(x,y1), hold off
sylabel('\Delta/h(h/d_{50})^{0.3}')
subplot(212), semilogy(T,y2,'o'), hold on
```

```
y2 = 0.015.*(1-exp(-0.5.*x)).*(25-x);
semilogy(x,y2), hold off
sxlabel('Transport stage parameter T')
sylabel('\Delta/\Lambda(h/d_{50})^{0.3}')
subplot

% (c)
u_star = sqrt(9.81.*h.*12.5e-5);
k_s = 12.*h.*exp(-0.4.*V./u_star) - 3.*d_90;

x = Delta./Lambda;
y = k_s./Delta;

figure(2)
semilogx(x,y,'o'), hold on
x = logspace(-3,log10(0.2),100);
y = 1.1.*(1-exp(-25.*x));
semilogx(x,y), hold off
sxlabel('Bedform steepness \Delta/\Lambda')
sylabel('k_s/\Delta')
```

Computer Problem 8.2

Consider the bedform and resistance to flow in the backwater profile analyzed in Computer Problem 3.1.

(a) Assume the rigid boundary is replaced with uniform 1-mm sand. Select an appropriate bedform predictor and determine the type of bedform to be expected along the 25-km reach of the channel using previously calculated hydraulic parameters. (Check your results with a second bedform predictor.) Also determine the corresponding resistance to flow along the channel and recalculate the backwater profile. Briefly discuss the methods, assumptions, and results. Three sketches should be provided along the 25 km of the reach: (i) type of bedforms; (ii) Manning n or Darcy-Weisbach f; and (iii) water surface elevation.

(b) Repeat the procedure described in (a) after replacing the bed material with the sediment size distribution from Problem 2.4.

Solution BASIC EQUATION OF BACKWATER SURFACE

The basic equation of backwater surface is

$$\frac{dh}{dx} = -S_0 \frac{1 - \left(\dfrac{h_n}{h}\right)^3}{1 - \left(\dfrac{h_c}{h}\right)^3} \tag{8.5}$$

in which the negative sign "-" means the direction of x direction toward upstream. h_c is estimated by

$$h_c = \left(\frac{q^2}{g}\right)^{\frac{1}{3}} \tag{8.6}$$

For alluvial channels, h_n is a function of **bedform** while bedform type in turn depends on flow depth h and other flow and sediment variables, i.e., $h_n = f(h, \text{others})$. For this case, we use Engelund's method to determine h_n. That is, solving the above two equations and the following equations simultaneously.

Form drag:

$$\begin{array}{rl} \text{If } \mathbf{Fr} < 1 \text{ and } \tau'_* < 0.06 & \tau'_* = \tau_* \\ \text{If } \mathbf{Fr} < 1 \text{ and } 0.06 \le \tau'_* < 0.55 & \tau'_* = 0.06 + 0.4\tau_*^2 \\ \text{If } \mathbf{Fr} \ge 1 \text{ and } 0.55 \le \tau'_* < 1 & \tau'_* = \tau_* \\ \text{If } \mathbf{Fr} \ge 1 \text{ and } 1 \le \tau'_* & \tau'_* = (0.702\tau_*^{-1.8} + 0.298)^{-1./1.8} \end{array}$$

Skin friction: $$\frac{V}{\sqrt{gh'S_0}} = 2.5 \ln\left(\frac{h'}{2.5D_s}\right) + 6$$

Continuity: $$Q = V h_n B$$

The plots are shown in the following page. A MATLAB program is shown in Appendix I.

RESULTS AND DISCUSSIONS

Bedform Type

Bedform type has been automatically considered in the processes of the calculations using Engelund method. But Engelund method can only discern lower flow regime and higher flow regime. To tell ripple and dune, Simons and Richardson's method is applied. Accordingly, the stream power ($\tau_0 V = \gamma h S_e V$, in which $S_e = S_0 + \dfrac{dh}{dx}$ in this case) is also listed in Table 1. Then the bedform type can be determined by reading Figure 8.8 (p.143) where $d_s = 0.3$ mm. The results are shown in the last column in Table 1 and Figure 1. It can be found that the bedform is initially ripple, and gradually becomes dune, transition and higher flow regime with the increase of stream power.

Water Surface Profile

The water surface is calculated using the present program and shown in Table 1 and in Figure 2. The flow depth at $x = 15000$ m is 2 m which is deeper than that in the rigid bed where $h = 1.74$ m. This is because the downstream dune resistance raises the water elevation. However, the flow depth far away from the dam is less than that in the rigid bed. This is because the higher flow regime resistance is smaller than that in rigid bed.

Darcy-Weisbach Friction Coefficient

The Darcy-Weisbach friction coefficient f is calculated by

$$f = \frac{8\tau_0}{\rho V^2} = \frac{8ghS_e}{V^2}$$

which is shown in Table 1 and Figure 3. It can be found that the maximum values of f occur between x = 4000 - 6500 m where the corresponding bedforms are dunes.

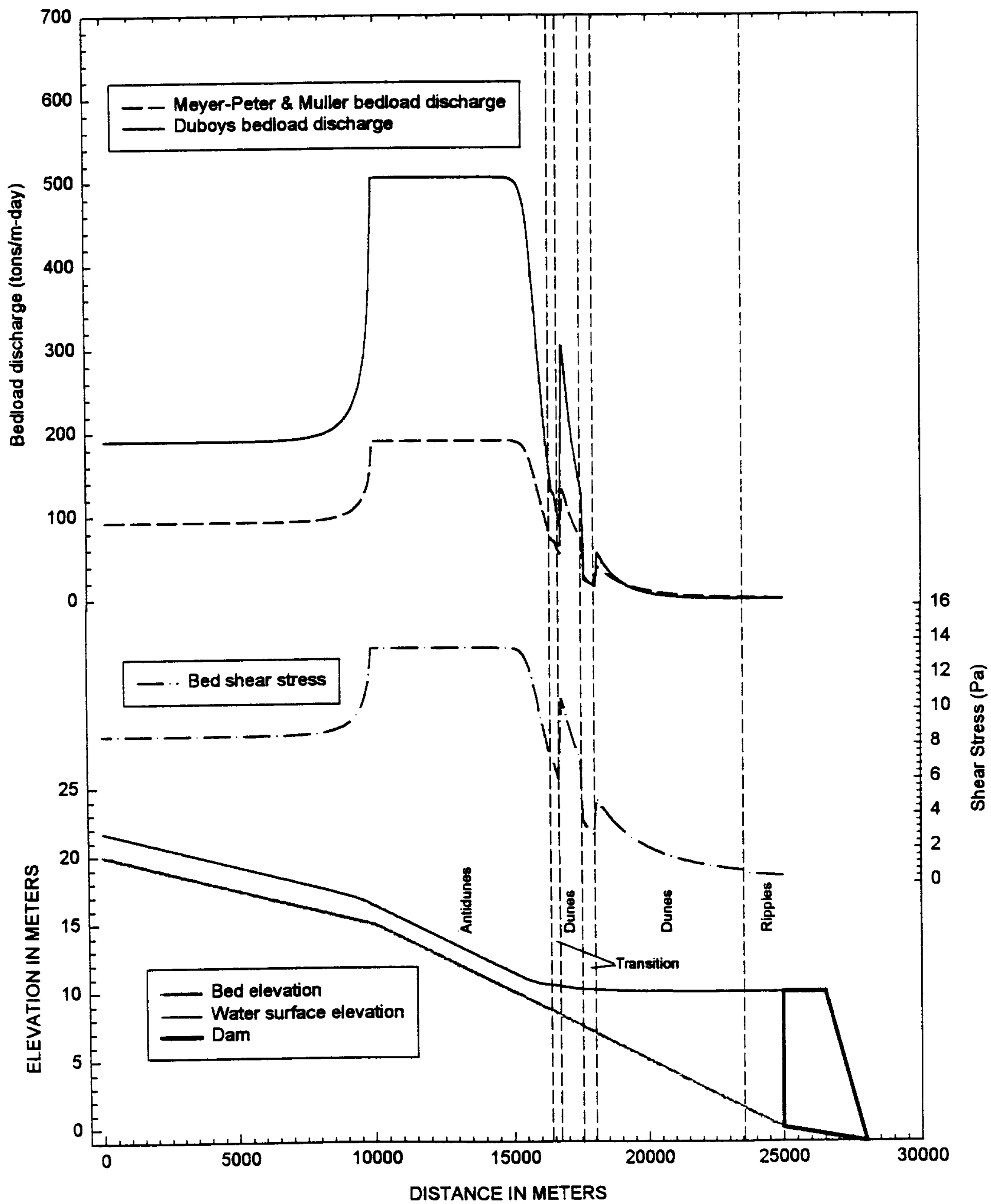


Table 1 Results of Computation

Distance from dam x (m)	Flow depth h (m)	Flow velocity V (m)	Darcy friction factor f	Stream power (tau)V (bl/ft.s)	Bedform type
0	10.000	0.372	0.038	0.017	Ripple
117	9.884	0.376	0.038	0.020	Ripple
1055	8.954	0.415	0.043	0.027	Ripple
1992	8.026	0.463	0.044	0.038	Ripple
2930	7.103	0.524	0.045	0.057	Ripple
3867	6.187	0.601	0.046	0.089	Dune
4805	5.283	0.704	0.048	0.150	Dune
5742	4.402	0.845	0.050	0.255	Dune
6497	3.724	0.999	0.049	0.403	Dune
7038	3.270	1.138	0.047	0.584	Transition
7472	2.937	1.267	0.046	0.796	Antidune & Plane
7835	2.690	1.383	0.046	1.029	Antidune & Plane
8150	2.505	1.485	0.045	1.272	Antidune & Plane
8435	2.365	1.573	0.045	1.520	Antidune & Plane
8723	2.252	1.652	0.046	1.756	Antidune & Plane
8990	2.172	1.712	0.045	1.956	Antidune & Plane
9250	2.116	1.758	0.045	2.119	Antidune & Plane
9514	2.075	1.792	0.046	2.246	Antidune & Plane
9790	2.047	1.817	0.046	2.340	Antidune & Plane
10090	2.028	1.834	0.046	2.407	Antidune & Plane
10426	2.015	1.846	0.046	2.451	Antidune & Plane
10814	2.007	1.853	0.045	2.477	Antidune & Plane
11280	2.003	1.857	0.045	2.491	Antidune & Plane
11864	2.001	1.859	0.045	2.498	Antidune & Plane
12643	2.000	1.860	0.045	2.500	Antidune & Plane
13581	2.000	1.860	0.045	2.501	Antidune & Plane
14518	2.000	1.860	0.045	2.501	Antidune & Plane
15000	2.000	1.860	0.045	2.501	Antidune & Plane
15000	2.000	1.860	0.054	1.250	Antidune & Plane
15078	2.000	1.860	0.023	1.250	Antidune & Plane
15703	2.000	1.860	0.023	1.250	Antidune & Plane
16328	2.000	1.860	0.023	1.250	Antidune & Plane
16953	2.000	1.860	0.023	1.250	Antidune & Plane
17578	2.000	1.860	0.023	1.250	Antidune & Plane
18203	2.000	1.860	0.023	1.250	Antidune & Plane
18828	2.000	1.860	0.023	1.250	Antidune & Plane
19453	2.000	1.860	0.023	1.250	Antidune & Plane

20078	2.000	1.860	0.023	1.250	Antidune & Plane
20703	2.000	1.860	0.023	1.250	Antidune & Plane
21328	2.000	1.860	0.023	1.250	Antidune & Plane
21953	2.000	1.860	0.023	1.250	Antidune & Plane
22578	2.000	1.860	0.023	1.250	Antidune & Plane
23203	2.000	1.860	0.023	1.250	Antidune & Plane
23828	2.000	1.860	0.023	1.250	Antidune & Plane
24453	2.000	1.860	0.023	1.250	Antidune & Plane
25000	2.000	1.860	0.023	1.250	Antidune & Plane

APPENDIX MATLAB PROGRAM

```
%Main Program: Computer Problem 8.2
%General parameter
q = 3.72; %Unit discharge (m^2/s)
rho = 1000; %Water density
g = 9.81;

%Plot the channel bed profile
x = linspace(0,15000,70); y = 0.001.*x;
plot(x,y), text(x,y-0.45,'\')
hold on

%Initial condition
x0 = 0; h0 = 10; xfinal = 15000;

S0 = 0.001;
[x,h1] = ode23v('profile',x0,xfinal,h0,S0,0);
v1 = q./h1;

for i =1:size(x)-1
    Se(i) = S0 + (h1(i+1)-h1(i))./(x(i+1)-x(i));
    tau(i) = rho.*g.*h1(i).*Se(i);
    f(i) = 8.*tau(i)./rho./v1(i).^2;
end
    f = [0.038 f]';
plot(x,0.001.*x+h1,'r--') %Plot water surface profile
plot(x,0.001.*x+f,'b:') %Plot shear stress

table1 = [x'; h1'; v1'; f']';
legend('channel bed','flow depth','velocity','shear stress')

%Plot the channel bed profile
```

```
x2 = linspace(15000,25000,50); y = 15 + (x2-15000).*0.0005;
plot(x2,y), text(x2,y-0.45,'\')

%Initial condition
x0 = 15000; h0 = h1(28); xfinal = 25000;

S0 = 0.0005;
[x3,h2] = ode23v('profile',x0,xfinal,h0,S0,0);
v2 = q./h2;

for i =1:size(x3)-1
    Se(i) = S0;
    tau(i) = rho.*g.*h2(i).*Se(i);
    ff(i) = 8.*tau(i)./rho./v2(i).^2;
end
    ff = [0.0454 ff]';
plot(x3,15+(x3-15000).*0.0005+h2,'r--')
plot(x3,15+(x3-15000).*0.0005+ff,'b:')

table2 = [x2'; h2'; v2'; ff']';
table = [table1;table2];

fid = fopen('cp3_1a.txt','w')
fprintf(fid,'%10.4f %10.4f %10.4f %10.4f\n',table')

title('Plot of Results of Computer Problem 3.1')
sxlabel('Distance from the dam x (m)')
sylabel('h(m), V(m/s), and \tau (N/m^2)')

stext(15000,8,'Note: The values of h, V and {\tau} are')
stext(15000,6,' relative to the local bed elevation')

%Backwater Surface Profile Equation
function dh = profile(x,h,S0,Se)
q = 3.72;
g = 9.81;
ds = 0.3e-3;
Se = S0 + dh;  %The x-aix direction is tarward to upstream
table = eng(q,1,ds,Se);
hn = table(1);
hc = (q.^2./g).^(1./3);
dh = -S0.*(1-(hn./h).^3)./(1-(hc./h).^3);function table = eng(Q,B,ds,Se)
```

```
function table = engelund(Q,B,ds,Se)
% [h,h1,V] = engelund(Q,B,ds,Se): Calculate flow depth
%            using Engelund method
%
% h = total hydraulic radius;
% h1 = sand hydraulic radius;
% V = mean velocity in m/s;
% Q = discharge in m^3/s;
%  B = channel width in m;
% ds = sediment size in m;
% Se = energy slope.
%
% Copy right by JUNKE GUO, Nov. 12, 1996

h = 1; %assumed flow depth
hh = 3; %another assumed flow depth. MATLAB while.
G = 2.65;
tau1 = 0.05;
Fr = 0.2;
while abs(hh-h) > 1e-4
      h = (hh+h)./2;
      tau = h.*Se./(G-1)./ds;

%Lower flow regime
      if Fr < 1
       if tau1 < 0.05
         tau1 = tau;
       else
         tau1 = 0.06+0.4.*tau.^2;
       end
      end

%Higher flow regime
      if Fr >= 1
        if tau1 >= 0.55 & tau1 < 1
         tau1 = tau;
        else
         tau1 = (0.702.*tau.^(-1.8)+0.298).^(-1./1.8);
        end
      end

      h1 = tau1.*(G-1).*ds./Se;
      V = sqrt(9.81.*h1.*Se).*2.5.*log(5.5.*h1./ds);
```

```
      hh = Q./V./B;
      Fr = V./sqrt(9.81.*hh);
end

table = [h h1 V];
```

Chapter 9

Bedload

Problem 9.1

Calculate the unit bedload discharge for a channel given the slope $S_0 = 0.01$, the flow depth $h = 20$ cm, and the grain size $d_{50} = 15$ mm. From Duboys' equation, calculate q_{bw} in lb/ft-s, and q_{bv} in ft^2/s.

Solution

$$\tau_0 = \gamma h S_0 = (62.4)(0.2/0.3048)(0.01) = 0.409 \text{ lb/ft}^2$$

$$\begin{aligned} q_{bv} &= \frac{0.173}{d_s^{3/4}}\tau_0(\tau_0 - 0.0125 - 0.019 d_s) \\ &= \frac{0.173}{15^{3/4}}(0.409)(0.409 - 0.0125 - 0.019 \times 15) \\ &= 1.04 \times 10^{-3} \text{ ft}^2\text{/s} \end{aligned}$$

$$q_{bw} = 1.04 \times 10^{-3}\,\frac{\text{ft}^3}{\text{ft-s}} = 1.04 \times 10^{-3}\frac{\text{ft}^3}{\text{ft-s}}\left(\frac{2.65 \times 62.4 \text{ lb}}{\text{ft}^3}\right) = 0.17 \text{ lb/ft-s}$$

Problem 9.2

Use Meyer-Peter and Muller's method to calculate q_{bm} in kg/m-s and q_{bv} in m^2/s for the conditions given in Problem 9.1.

Solution

$$\frac{q_{bv}}{\sqrt{(G-1)\,g d_s^3}} = 8(\tau_* - \tau_{*c})^{1.5}$$

in which

$$\sqrt{(G-1)\,g d_s^3} = \sqrt{(1.65)(9.81)(15 \times 10^{-3})^3} = 0.00739 \text{ m}^2\text{/s}$$

$$\tau_* = \frac{\gamma h S_0}{(\gamma_s - \gamma)\,d_s} = \frac{h S_0}{(G-1)\,d_s} = \frac{(0.2)(0.01)}{(1.65)(0.015)} = 8.08 \times 10^{-2}$$

Therefore,

$$\begin{aligned} q_{bv} &= 8\sqrt{(G-1)\,gd_s^3}\,(\tau_* - \tau_{*c})^{1.5} \\ &= 8\,(0.00739)\,(0.0808 - 0.047)^{1.5} \\ &= 3.67 \times 10^{-4}\ \mathrm{m^2/s} \end{aligned}$$

Problem 9.3

Use Einstein and Brown's method to calculate the bedload transport rate in a 100-m-wide coarse sand-bed channel with slope $S_0 = 0.003$ when the applied shear stress τ_0 equals τ_c. Determine the transport rate Q_{bv} in $\mathrm{m^3/s}$ and in $\mathrm{ft^3/s}$.

Solution When $\tau_0 = \tau_c$, we have

$$\tau_* = 0.047 < 0.18$$

Then Equation (9.5a) should be used.

$$\frac{q_{bv}}{\omega_0 d_s} = 2.15 e^{-0.391/\tau_*}$$

in which for coarse sand, we have $d_s = 0.75$ mm.

$$\begin{aligned} \omega_0 &= \sqrt{(G-1)\,gd_s}\left(\sqrt{\frac{2}{3} + \frac{36\nu^2}{(G-1)\,gd_s^3}} - \sqrt{\frac{36\nu^2}{(G-1)\,gd_s^3}}\right) \\ &= \sqrt{(1.65)\,(9.81)\,(0.75\times10^{-3})}\left(\sqrt{\frac{2}{3} + \frac{36\,(1.0\times10^{-6})^2}{(1.65)\,(9.81)\,(0.75\times10^{-3})^3}}\right. \\ &\quad \left. - \sqrt{\frac{36\,(1.0\times10^{-6})^2}{(1.65)\,(9.81)\,(0.75\times10^{-3})^3}}\right) \\ &= 0.0823\ \mathrm{m/s} \end{aligned}$$

Therefore,

$$\begin{aligned} q_{bv} &= 2.15\omega_0 d_s e^{-0.391/\tau_*} \\ &= 2.15\,(0.0823)\left(0.75\times10^{-3}\right) e^{-0.391/0.047} \\ &= 3.24\times10^{-8}\ \mathrm{m^2/s} \end{aligned}$$

and

$$\begin{aligned} Q_{bv} &= Wq_{bv} = (100)\left(3.24\times10^{-8}\right) = 3.24\times10^{-6}\ \mathrm{m^3/s} \\ &= 1.14\times10^{-4}\ \mathrm{ft^3/s} \end{aligned}$$

Problem 9.4

Use Einstein and Brown's bedload equation to calculate q_{bm} in kg/m-s and q_{bw}, in lb/ft-s for the conditions given in Problem 9. 1.

Solution

$$\tau_* = \frac{\tau_0}{(\gamma_s - \gamma)\, d_{50}} = \frac{hS_0}{(G-1)\, d_{50}} = \frac{(0.2)\,(0.01)}{(1.65)\,(0.015)} = 0.0808$$

$$\omega_0 = 0.4 \text{ m/s}$$

$$q_{bv} = \omega_0 d_{50}\,(2.15)\, e^{-0.391\tau_*} = 1.02 \times 10^{-4} \text{ m}^2/s$$

$$q_{bm} = \rho_s q_{bv} = \left(2650 \frac{\text{kg}}{\text{m}^3}\right) 1.02 \times 10^{-4}\ \frac{\text{m}^2}{\text{s}} = 0.27 \text{ kg/m-s}$$

$$q_{bw} = \gamma_s q_{bv} = (2.65 \times 62.4)\left(1.02 \times 10^{-4}/0.3048^2\right) = 0.18 \text{ lb/ft-s}$$

Problem 9.5

Use the three methods detailed in this chapter to calculate the daily bedload in metric tons in a 20-m-wide medium gravel-bed canal with a slope $S_0 = 0.001$ and at flow depth $h = 2$ m. Compare the results in metric tons per day.

Solution Duboys' equation

$$q_{bv} = \frac{0.173}{d_s^{3/4}} \tau_0 \left(\tau_0 - 0.0125 - 0.019 d_s\right)$$

in which q_{bv} in ft^2/s, d_s in mm, and τ_0 in lb/ft^2.

For this case, $d_s = 12$ mm (medium gravel), $h = 2$ m $= 6.56$ ft, $W = 20$ m $= 65.6$ ft,

$$\tau_0 = \gamma h S_0 = (62.4)\,(6.56)\,(0.001) = 0.409 \text{ lb/ft}^2$$

Thus,

$$q_{bv} = \frac{0.173}{12^{0.75}}\,(0.409)\,(0.409 - 0.0125 - 0.019 \times 12) = 1.84 \times 10^{-3} \text{ ft}^2/\text{s}$$

$$\begin{aligned} Q_{bv} &= q_{bv} W = 1.84 \times 10^{-3} \frac{\text{ft}^2}{\text{s}}\,(65.6)\ \text{ft} = 0.121 \text{ ft}^3/\text{s} \\ &= (0.121)\,\frac{\text{ft}^3}{\text{s}}\,(0.3048)^3\,\frac{\text{m}^3}{\text{ft}^3}\,(3600 \times 24)\ \frac{\text{s}}{\text{day}} \\ &= 296 \text{ m}^3/\text{day} \end{aligned}$$

$$Q_{bm} = \rho_s Q_{bv} = (2.65)\,\frac{\text{metric tons}}{\text{m}^3}\,(296)\,\frac{\text{m}^3}{\text{day}} = 786 \text{ metric tons/day}$$

Meyer-Peter and Muller's equation

$$\frac{q_{bv}}{\sqrt{(G-1)\,gd_s^3}} = 8\left(\tau_* - \tau_{*c}\right)^{3/2}$$

For this case, $d_s = 12$ mm, From Table 7.1

$$\tau_{*c} = 0.047$$

$$\tau_* = \frac{hS_0}{(G-1)\,d_s} = \frac{(2)\,(0.001)}{(1.65)\,(0.012)} = 0.101$$

Thus,

$$q_{bv} = 8\,(0.101 - 0.047)^{1.5}\sqrt{(1.65)\,(9.81)\,(0.012)^3} = 5.31 \times 10^{-4}\ \mathrm{m^2/s}$$

$$Q_{bv} = Wq_{bv} = (20)\left(5.31 \times 10^{-4}\right) = 0.0106\ \mathrm{m^3/s} = 917\ \mathrm{m^3/day}$$

$$Q_{bm} = \rho_s Q_{bv} = (2.65)\,(917) = 2431 \text{ metric tons/day}$$

Einstein and Brown's equation

$$\frac{q_{bv}}{\omega_0 d_s} = \begin{cases} 2.15e^{-0.391/\tau_*} & \tau_* < 0.18 \\ 40\tau_*^3 & 0.18 \le \tau_* < 0.52 \\ 15\tau_*^{1.5} & 0.52 \le \tau_* \end{cases}$$

in which $\omega_0 =$ Rubey's fall velocity.

For $d_s = 12$ mm and $T = 20°$C, we have

$$\omega_0 = 0.359\ \mathrm{m/s}$$

From the previous result, $\tau_* = 0.101 < 0.18$, thus

$$\frac{q_{bv}}{\omega_0 d_s} = 2.15e^{-0.391/\tau_*} = 2.15e^{-\dfrac{0.391}{0.101}} = 0.0447$$

$$q_{bv} = (0.0447)\,(0.259)\,(0.012) = 0.000192\ \mathrm{m^2/s}$$

$$Q_{bv} = \left(1.92 \times 10^{-4}\right)(20) = 0.0038\ \mathrm{m^3/s} = 332\ \mathrm{m^3/day}$$

$$Q_{bm} = \rho_s Q_{bv} = (2.65)\,(332) = 881 \text{ metric tons/day}$$

Problem 9.6

(a) Which bed sediment sampler would you recommend for the canal in Problem 9.5? (b)Which method would you recommend for measuring and controlling bedload in the same canal?

Solution (a) A bucket sampler; and (b) A Helley-Smith sampler to measure bedload. Reducing the flow depth to about 1 m would reduce bedload.

Problem 9.7

With reference to the Bergsche Maas bedform data given in Computer Problem 8.1, (a) calculate the bedload sediment transport; and (b) estimate the time required for the bedload to fill the volume of a representative dune.

Solution (a) Meyer-Peter and Muller's method:

$$\frac{q_{bv}}{\sqrt{(G-1)\,g d_s^3}} = 8\left(\tau_* - \tau_{*c}\right)^{3/2}$$

in which

$$\tau_{*c} = \begin{cases} 0.5\tan\phi & \text{when } d_* < 0.3 \\ 0.25 d_*^{-0.6}\tan\phi & \text{when } 0.3 \le d_* < 19 \\ 0.013 d_*^{0.4}\tan\phi & \text{when } 19 \le d_* < 50 \\ 0.06\tan\phi & \text{when } d_* \ge 50 \end{cases}$$

and $\phi = 30°$. The results are shown below.

Q (m^3/s)	S_f (cm/km)	h (m)	V (m/s)	d_s (μm)	Δ (m)	Λ (m)	T	C' ($m^{1/2}/s$)	q_{bv} (10^{-3}) m^2/s
2160.00	12.50	8.60	1.35	480	1.50	22.50	8.2	86.8	0.5163
2160.00	12.50	8.00	1.35	410	1.00	14.00	9.6	87.6	0.4630
2160.00	12.50	10.50	1.30	300	1.50	30.00	11.3	91.8	0.7047
2160.00	12.50	10.00	1.40	500	1.60	32.00	8.4	87.4	0.6503
2160.00	12.50	7.60	1.40	520	1.40	21.00	8.5	85.5	0.4263
2160.00	12.50	8.40	1.40	380	1.50	22.50	11.1	88.5	0.4996
2160.00	12.50	8.70	1.70	300	1.50	30.00	20.6	90.5	0.5288
2160.00	12.50	7.50	1.55	250	2.50	50.00	20.3	91.0	0.4223
2160.00	12.50	8.30	1.50	260	1.80	36.00	18.0	91.3	0.4929
2160.00	12.50	9.50	1.35	230	1.80	36.00	15.7	93.2	0.6065
2160.00	12.50	8.80	1.35	240	1.80	36.00	15.3	92.4	0.5395
2160.00	12.50	9.00	1.30	240	1.80	36.00	14.1	92.6	0.5583
2160.00	12.50	9.60	1.50	220	2.20	33.00	20.5	93.6	0.6166
2160.00	12.50	8.70	1.50	370	1.90	28.50	13.1	88.9	0.5274
2160.00	12.50	8.20	1.35	330	2.00	36.00	11.7	89.5	0.4824
2160.00	12.50	8.20	1.35	480	1.40	22.40	8.3	86.5	0.4798
2160.00	12.50	8.10	1.40	350	1.00	20.00	12.0	88.9	0.4730
2160.00	12.50	8.00	1.50	420	0.60	9.00	11.9	87.4	0.4629
2160.00	12.50	7.80	1.50	410	0.60	9.00	12.2	87.4	0.4454
2160.00	12.50	6.80	1.50	400	0.40	8.00	12.8	86.6	0.3607
2160.00	12.50	6.40	1.50	270	0.60	6.00	18.2	89.3	0.3306
2160.00	12.50	5.80	1.50	220	1.00	10.00	22.1	90.2	0.2850
2160.00	12.50	6.20	1.50	210	1.20	24.00	22.7	91.0	0.3161
2160.00	12.50	6.60	1.50	210	1.00	50.00	22.5	91.5	0.3479
2160.00	12.50	8.30	1.50	180	0.90	36.00	24.9	94.2	0.4949
2160.00	12.50	8.10	1.35	400	1.50	22.50	9.8	87.9	0.4721

Source: After Julien (1992)

(b) The average bedload rate is $q_{bv} = 4.81 \times 10^{-4}$ m^2/s, the average dune length is $\Lambda = 26.2$ m, the average dune height is $\Delta = 1.38$ m, so the average time to fill a representative dune is

$$t = \frac{0.5\Lambda\Delta}{q_{bv}} = \frac{0.5\,(26.2)\,(1.38)}{4.81 \times 10^{-4}} = 37674 \text{ s} = 10.5 \text{ hrs}$$

MatLab Program

```
% Problem 9.7
A = [2160.00 12.50    8.60   1.35 480 1.50 22.50
    2160.00 12.50    8.00   1.35 410 1.00 14.00
    2160.00 12.50   10.50   1.30 300 1.50 30.00
    2160.00 12.50   10.00   1.40 500 1.60 32.00
    2160.00 12.50    7.60   1.40 520 1.40 21.00
    2160.00 12.50    8.40   1.40 380 1.50 22.50
    2160.00 12.50    8.70   1.70 300 1.50 30.00
    2160.00 12.50    7.50   1.55 250 2.50 50.00
    2160.00 12.50    8.30   1.50 260 1.80 36.00
    2160.00 12.50    9.50   1.35 230 1.80 36.00
    2160.00 12.50    8.80   1.35 240 1.80 36.00
    2160.00 12.50    9.00   1.30 240 1.80 36.00
    2160.00 12.50    9.60   1.50 220 2.20 33.00
    2160.00 12.50    8.70   1.50 370 1.90 28.50
    2160.00 12.50    8.20   1.35 330 2.00 36.00
    2160.00 12.50    8.20   1.35 480 1.40 22.40
    2160.00 12.50    8.10   1.40 350 1.00 20.00
    2160.00 12.50    8.00   1.50 420 0.60  9.00
    2160.00 12.50    7.80   1.50 410 0.60  9.00
    2160.00 12.50    6.80   1.50 400 0.40  8.00
    2160.00 12.50    6.40   1.50 270 0.60  6.00
    2160.00 12.50    5.80   1.50 220 1.00 10.00
    2160.00 12.50    6.20   1.50 210 1.20 24.00
    2160.00 12.50    6.60   1.50 210 1.00 50.00
    2160.00 12.50    8.30   1.50 180 0.90 36.00
    2160.00 12.50    8.10   1.35 400 1.50 22.50];

S=A(:,2).*1e-5; h=A(:,3); V=A(:,4); d_s=1e-6*A(:,5); Delta=A(:,6); Lambda=A(:,7);

% (a) bedload rate
tau = h.*S./(1.65.*d_s); % nondimensional

d_star = d_s.*(1.65*9.81/1e-12).^(1/3);
phi = radian(30);
```

```
if d_star < 0.3, tau_c = 0.5*tan(phi); % nondimensional
elseif d_star >=0.3 & d_star < 19, tau_c = 0.25.*d_star.^(-0.6).*tan(phi);
elseif d_star >= 19 and d_star < 50, tau_c =0.013.*d_star^0.4*tan(phi);
else tau_c = 0.06*tan(phi);

q_bv = sqrt(1.65*9.81.*d_s.^3)*8.*(tau - tau_c).^1.5; %unit: m^2/s

% (b)
q_bv = mean(q_bv);
Lambda = mean(Lambda); Delta = mean(Delta);
Area = 0.5*Lambda*Delta; % filled area
Time = Area/q_bv;  % unit: s
Time = Time/3600;  % unit: hour
```

Computer Problem 9.1

Consider the channel reach analyzed in Computer Problems 3.1 and 8.2. Select an appropriate bedload relationship to calculate the bed sediment discharge in tons /m-day for the uniform 1-mm sand in Computer Problem 8.2. Plot the results along the 25-km reach and discuss the method, assumptions, and results.

Solution Duboy's equation and Meyer-Peter amd Muller's equation are used. The results are shown in the figure of Computer Problem 8.2.

Computer Problem 9.2

Write a computer program to calculate the bedload transport rate by size fraction from the methods of Duboys, Meyer-Peter and Muller, and Einstein and Brown, and repeat the calculations of the tabulation in Case Study 9.1 at $h = 0.4$ ft and $\tau_0 = 0.04$ lb/ft^2.

Solution The program is written in MatLab as below. The results (with Case Study 9.1) are shown in the table on next page.

```
% Computer Problem 9.2
d_s = [0.074 0.125 0.246 0.351 0.495 0.700 0.990 1.400...
       1.980 3.960]; % mm
dp_i = [0.002 0.00815 0.02935 0.0865 0.1555 0.2075 0.21050 0.14950...
        0.0995 0.0515];
tau_0 = 0.06.*ones(size(d_s)); % lb/ft^2

q1_bv = duboy(d_s,tau_0).*dp_i; %lb/ft-s
q2_bv = meyer(d_s,tau_0).*dp_i; %lb/ft-s

T = 25.6; tau_star = tau_0./1.65./62.4./(d_s./1000./0.3048);
```

```
q3_bv = brown(d_s,tau_star).*dp_i; %lb/ft-s

A = [d_s; dp_i; q1_bv*1e6; q2_bv*1e6; q3_bv*1e6]'
fid = fopen('cp9_2.txt','w');
fprintf(fid,'%5.3f& %8.4f& %7.3f& %7.3f& %7.3f\\\\\n',A');

sum1a = sum(q1_bv), sum1b = 2.65*62.4*sum1a,
sum2a = sum(q2_bv), sum2b = 2.65*62.4*sum2a,
sum3a = sum(q3_bv), sum3b = 2.65*62.4*sum3a,

function q_bv = brown(d_s,tau_star)
% d_s: mm; tau_star: nondimensional; q_bv: m^2/s
if tau_star < 0.18, q_bvs = 2.15.*exp(-0.391./tau_star);
elseif tau_star >= 0.52, q_bvs = 15.*tau_star.^1.5;
else, q_bvs = 40.*tau_star.^3; end

d_s = d_s.*1e-3; nu = 1e-5*0.3048.^2;
w = (sqrt(10.791.*d_s.^3 + 36.*nu.^2) - 6.*nu)./d_s;

q_bv = q_bvs.*w.*d_s./0.3048.^2;        % unit: ft^2/s

function q_bv = duboy(d_s, tau_0)
% d_s: mm; tau_0: lb/ft^2; q_bv: ft^2/s
q_bv = 0.173./d_s.^0.75.*tau_0.*(tau_0 - 0.0125 - 0.019.*d_s);
n = size(d_s); n = n(2);
for i = 1:n,
    if q_bv(i) < 0, q_bv(i) = 0; end
end

function q_bv = meyer(d_s, tau_0)
% tau_0: lb/ft^2; d_s: mm
d_s = d_s/1000/0.3048; tau_c = 5.*d_s;  % psf

n = size(d_s); n = n(2);
for i = 1:n
 if tau_0(i) > tau_c(i),
    q_bv(i) = 12.9/165.4/sqrt(1000/515.4).*(tau_0(i) - tau_c(i)).^1.5;
 else
    q_bv(i) = 0;
 end
end
```

d_s (mm)	Δp_i	Duboy $q_{bvi}\Delta p_i$ (10^{-6} ft^2/s)	Meyer-Peter and Muller $q_{bvi}\Delta p_i$ (10^{-6} ft^2/s)	Einstein-Brown $q_{bvi}\Delta p_i$ (10^{-6} ft^2/s)
		$h = 0.2$ ft, $\tau_0 = 0.02$ lb/ft^2		
0.074	0.0020	0.297	0.288	0.167
0.125	0.0081	0.687	1.097	0.614
0.246	0.0294	0.822	3.315	1.457
0.351	0.0865	0.545	8.232	2.991
0.495	0.1555	0.000	11.274	3.555
0.700	0.2075	0.000	9.132	3.001
0.990	0.2105	0.000	2.717	1.879
1.400	0.1495	0.000	0.000	0.811
1.980	0.0995	0.000	0.000	0.325
3.960	0.0515	0.000	0.000	0.060
Total	$q_{bv} =$	2.35×10^{-6} ft^2/s	3.61×10^{-5} ft^2/s	1.48×10^{-5} ft^2/s
	$q_{bw} =$	3.89×10^{-4} lb/ft-s	0.006 lb/ft-s	0.0025 lb/ft-s
		$h = 0.4$ ft, $\tau_0 = 0.04$ lb/ft^2		
0.074	0.0020	2.545	0.855	1.338
0.125	0.0081	6.740	3.374	4.914
0.246	0.0294	13.272	11.208	11.659
0.351	0.0865	27.343	30.689	23.932
0.495	0.1555	32.994	49.560	28.441
0.700	0.2075	26.643	55.950	24.011
0.990	0.2105	12.754	43.166	15.029
1.400	0.1495	0.723	18.610	6.489
1.980	0.0995	0.000	3.633	2.602
3.960	0.0515	0.000	0.000	0.482
Total	$q_{bv} =$	1.23×10^{-4} ft^2/s	2.17×10^{-4} ft^2/s	1.19×10^{-4} ft^2/s
	$q_{bw} =$	0.0203 lb/ft-s	0.0359 lb/ft-s	0.0197 lb/ft-s
		$h = 0.6$ ft, $\tau_0 = 0.06$ lb/ft^2		
0.074	0.0020	6.744	1.596	4.517
0.125	0.0081	18.159	6.366	16.584
0.246	0.0294	37.352	21.757	39.349
0.351	0.0865	80.394	61.185	80.770
0.495	0.1555	104.194	102.886	95.989
0.700	0.2075	96.254	124.161	81.038
0.990	0.2105	63.162	107.893	50.723
1.400	0.1495	25.199	59.658	21.900
1.980	0.0995	6.113	25.434	8.780
3.960	0.0515	0.000	0.000	1.627
Total	$q_{bv} =$	4.37×10^{-4} ft^2/s	5.11×10^{-4} ft^2/s	4.01×10^{-4} ft^2/s
	$q_{bw} =$	0.0724 lb/ft-s	0.0845 lb/ft-s	0.0664 lb/ft-s

Chapter 10

Suspended load

Einstein sediment transport equation (Guo and Wood, 1995; Guo, Wu and Molinas, 1996)

$$q_s = du_* C_a \left(\frac{\xi_a}{1-\xi_a}\right)^z \int_{\xi_a}^{1} \left[\left(\frac{U}{u_*}+\frac{1}{\kappa}\right)+\frac{1}{\kappa}\ln\xi\right]\left(\frac{1-\xi}{\xi}\right)^z d\xi$$

which can be written as

$$q_s = du_* C_a \left(\frac{\xi_a}{1-\xi_a}\right)^z \left[\left(\frac{U}{u_*}+\frac{1}{\kappa}\right) J_1(z) + \frac{1}{\kappa} J_2(z)\right]$$

in which

$$\begin{aligned} J_1(z) &= \int_{\xi_a}^{1} \left(\frac{1-\xi}{\xi}\right)^z d\xi \\ J_2(z) &= \int_{\xi_a}^{1} \left(\frac{1-\xi}{\xi}\right)^z \ln\xi d\xi \end{aligned}$$

The integrals J_1 and J_2 can be calculated as follows:
For $z < 1$,

$$\begin{aligned} J_1(z) &= \frac{z\pi}{\sin z\pi} - \frac{E^{1-z}}{1-z} \\ J_2(z) &= -\frac{z\pi}{\sin z\pi} f(z) - \frac{E^{1-z}\ln E}{1-z} + \frac{E^{1-z}}{(1-z)^2} \end{aligned}$$

in which

$$f(z) = 0.4227843351 - \frac{1}{2\beta}\ln\frac{\Gamma(1-z+\beta)}{\Gamma(1-z-\beta)}$$

where β is a very small number, for example 10^{-5}. $f(z)$ can be approximated as

$$f(z) \approx 0.4227843351 - \ln(2-z) + \frac{1}{1-z} + \frac{1}{2(2-z)} + \frac{1}{24(2-z)^2}$$

For $z = 1$, J_1 and J_2 are calculated by

$$J_1(z=1) = -\ln E + E - 1$$
$$J_2(z=1) = 1 + E\ln E - E - \frac{1}{2}(\ln E)^2$$

For $z > 1$, the following recurrence formulas are used.

$$J_1(z) = \frac{1}{z-1}\left[\frac{(1-E)^z}{E^{z-1}} - zJ_1(z-1)\right]$$
$$J_2(z) = \frac{1}{z-1}\left[\frac{(1-E)^z}{E^{z-1}}\ln E - zJ_2(z-1) + J_1(z)\right]$$

References:

Guo, J. and Wood, W. L. (1995). “Fine Suspended Sediment Transport Rates.” *J. of Hydr. Engrg.,* **ASCE**, 121(12), 919-922.

Guo, J., Wu, B., and Molinas, A. (1996). “Analytical Investigation of Einstein's Sediment Transport Integrals.” *Proc. of International Conference on Reservoir Sedimentation*, Vol. 1, Edited by M. L. Albertson, A. Molinas, and R. Hotchkiss, Colorado State University, Fort Collins, CO, 119-124.

Exercise

1. Derive concentration by weight C_w, [Eq. (10.1b)] from concentration by volume C_v, (Eq. (10.1a)] given the density of sediment particles $G = \gamma_s/\gamma$.

   Solution

$$C_W = \frac{\text{sediment weight}}{\text{total weight}} = \frac{\rho_s V_s}{\rho V_w + \rho_s V_s} = \frac{\rho_s V_s}{\rho(V_t - V_s) + \rho_s V_s}$$
$$= \frac{\rho_s V_s}{\rho V_t + (\rho_s - \rho) V_s} = \frac{\frac{\rho_s}{\rho}\frac{V_s}{V_T}}{1 + \left(\frac{\rho_s}{\rho} - 1\right)\frac{V_s}{V_t}} = \frac{GC_v}{1 + (G-1)C_v}$$

2. (a) Derive Equation (10.19) from Equation (10.18); and (b) evaluate the maximum value of ε_z on a vertical.

   Solution (a)

$$\varepsilon_z = \beta_s \varepsilon_m = \beta_s \frac{\tau}{\rho_m}\frac{dz}{dv_x}$$

   in which

$$\tau = \tau_0\left(1 - \frac{z}{h}\right)$$
$$\frac{dv_x}{dz} = \frac{u_*}{\kappa z}$$

then

$$\begin{aligned}\varepsilon_s &= \beta_s \frac{\tau_0 \left(1 - \frac{z}{h}\right)}{\rho_m} \frac{\kappa z}{u_*} = \beta_s \kappa \frac{\tau_0}{\rho_m u_*} \frac{z}{h} (h - z) \\ &= \beta_s \kappa u_* \frac{z}{h} (h - z)\end{aligned}$$

in which $\tau_0 = \rho_m u_*^2$ is considered.

(b) The maximum value of ε_s occurs at $z = 0.5h$. Assume that $\beta_s = 1$and $\kappa = 0.4$, then

$$\varepsilon_{s\max} = (1)(0.4)(0.5)(0.5h)\, u_* = 0.1 h u_*$$

Problem 10.1

Plot the dimensionless concentration profiles C/C_a for medium silt, fine sand, and coarse sand in a 3-m-deep stream sloping at $S_0 = 0.002$. (Hint: Assume $a = 2d_s$)

Solution Concentration profile:

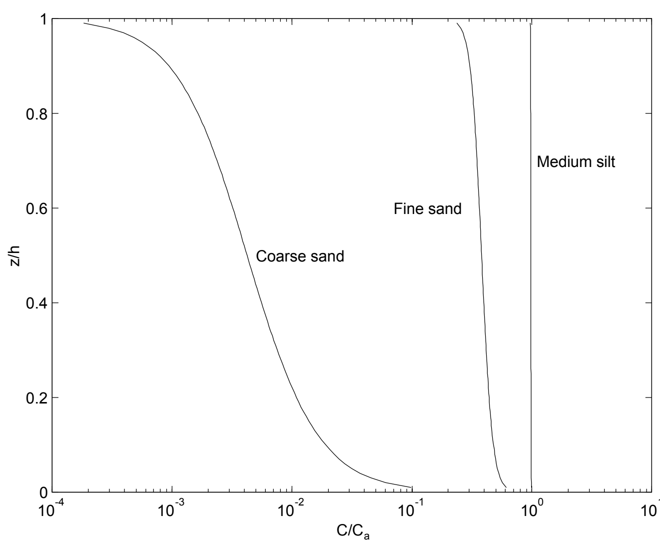
$$\frac{C}{C_a} = \left(\frac{1-\xi}{\xi} \cdot \frac{\xi_a}{1-\xi_a}\right)^{\mathrm{Ro}}$$

in which $\xi = z/h$, $\xi_a = a/h$, and Ro$= \omega / (\beta_s \kappa u_*)$. Assume that $T = 10°\mathrm{C}$, $\beta_s = 1$and $\kappa = 0.4$. $u_* = \sqrt{ghS_0} = \sqrt{(9.81)(3)(0.002)} = 0.243$ m/s.

	d_s (mm)	ξ_a	ω^a (mm/s)	Ro
Medium silt	0.016	1.07×10^{-5}	0.167	1.72×10^{-3}
Fine sand	0.125	8.33×10^{-5}	10.1	0.103
Coarse sand	0.5	3.33×10^{-4}	66.4	0.683

[a] From Table 5.4

The concentration profiles are shown in the following figure.

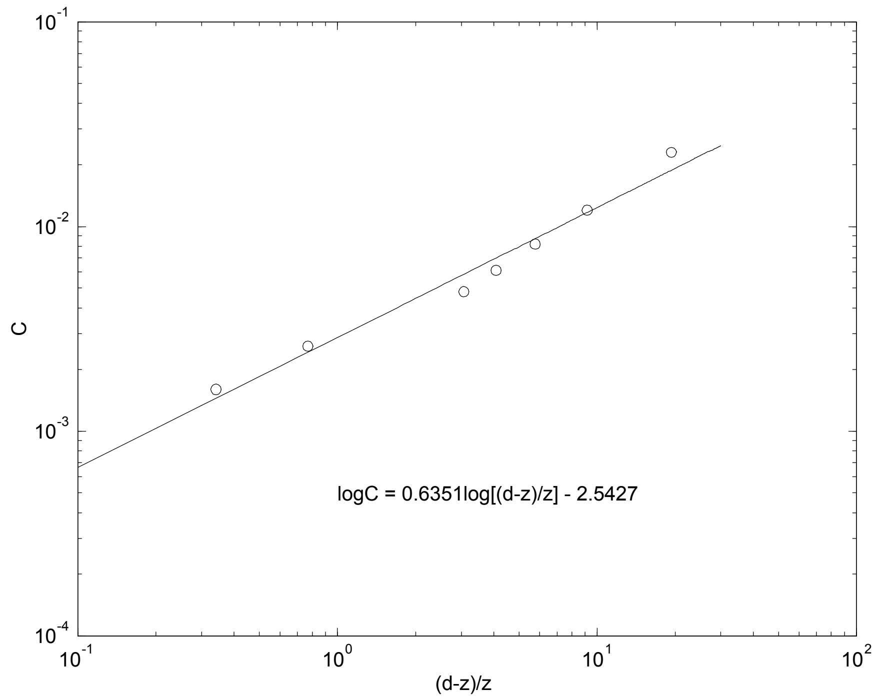


Problem 10.2

Given the sediment concentration profile from Problem 6.1:

(a) plot the concentration profile $\log C$ versus $log[(h-z)/z]$; (b) estimate the particle diameter from the Rouse number in (a); and (c) determine the unit sediment discharge from the given data.

Solution (a) Plot the concentration profile $\log C$ versus $log[(h-z)/z]$.



(b) The Rouse number is Ro= 0.6351.

(c) The unit sediment discharge can be calculated as follows:

C (10^{-3})	u (m/s)	z (mm)	q_s $10^{-4}(\mathrm{m}^2/\mathrm{s})$
23	0.734	6	1.01
12	0.789	12	1.14
8.2	0.827	18	1.22
6.1	0.867	24	1.27
4.8	0.891	30	1.28
2.6	0.987	69	1.77
1.6	1.030	91	1.50
0.76	1.048	122	0.972
$\sum$			1.02×10^{-3}

Hence, $q_s = 1.02 \times 10^{-3}$ m^2/s.

Problem 10.3

Calculate the daily sediment load in a nearly rectangular 50-m-wide stream with an average flow depth $h = 2$ m and slope $S_0 = 0.0002$ when 25% of the sediment load is fine silt, 25% is very fine silt, 25% is clay, and the middepth concentration is $C = 50,000$ mg/l.

Solution The shear velocity is

$$u_* = \sqrt{ghS_0} = \sqrt{(9.81)\,(2)\,(0.0002)} = 0.0626 \text{ m/s}$$

Assume $T = 10°$C, $\beta_s = 1$, and $\kappa = 0.4$, then we have

Sediment	d_s (mm)	ω (m/s)	Ro$= \omega/(\beta_s \kappa u_*)$	a (mm)
Clay	0.002	2.6×10^{-6}	1.04×10^{-4}	4×10^{-6}
Very fine silt	0.004	1×10^{-5}	3.99×10^{-4}	8×10^{-6}
Fine silt	0.008	4.2×10^{-5}	1.68×10^{-3}	16×10^{-6}

From the above table, we see that the Rouse numbers are very small. So, the concentration profiles are almost uniform, i.e., $C = 50,000$ mg/l$= 50$ kg/m^3 anywhere.

Assume the channel bed is smooth. The average velocity from equation (6.21) is

$$\begin{aligned} V &= \frac{u_*}{\kappa} \ln \frac{u_* h}{\nu} + 3.25 u_* \\ &= \frac{0.0626}{0.4} \ln \frac{(0.0626)\,(2)}{1.31 \times 10^{-6}} + (3.25)\,(0.0626) \\ &= 1.99 \text{ m/s} \end{aligned}$$

Then the sediment transport rate is

$$\begin{aligned} Q_s &= VWhC = (1.99)\,(50)\,(2)\,(50) = 9950 \text{ kg/s} \\ &= 9950 \,\frac{\text{kg}}{\text{s}} \left(\frac{1 \text{ ton}}{1000 \text{ kg}}\right) \left(\frac{3600 \times 24 \text{ s}}{\text{day}}\right) \\ &= 859,680 \text{ tons/day} \end{aligned}$$

Problem 10.4

A physical model of a river 50 m wide and 2 m deep is to be constructed in the hydraulics laboratory at a scale of 1: 100 horizontal and 1: 20 vertical. Calculate the ratio of transversal to vertical mixing time scales: (a) for the model and (b) for the prototype.

Solution (a) Then the transversal mixing timescale is $t_t = \frac{L_t}{V} = \frac{W^2}{hu_*}$. The vertical mixing timescale is $t_v = \frac{h}{V} = \frac{h}{0.1u_*}$. The ratio is

$$\frac{t_t}{t_v} = \frac{W^2}{hu_*} \cdot \frac{0.1u_*}{h} = 0.1 \left(\frac{W}{h}\right)^2$$

For the model, we have $W_m = 50/100 = 0.5$ m, $h_m = 2/20 = 0.1$ m. Then $t_t/t_v = 0.1\,(0.5/0.1)^2 = 2.5$.

For the prototype, we have $t_t/t_v = 0.1\,(50/2)^2 = 62.5$.

Problem 10.5

Calculate the distance required for complete transversal mixing in a large river at a discharge of 500,000 ft^3/s. Assume an average river width of 2 mi, a slope of about 0.4 ft/mi, and Manning coefficient $n = 0.02$.

Solution Given $Q = 500,000$ ft^3/s, $W = 2$ miles= 10560 ft, $S_0 = 4$ ft/mile= 7.576×10^{-5}, and $n = 0.02$, according to Manning equation, we have

$$h = \left(\frac{nQ}{1.49 S_0^{1/2} W}\right)^{0.6} = \left(\frac{(0.02)\,(500000)}{1.49\,(10560)\,\sqrt{7.576 \times 10^{-5}}}\right)^{0.6} = 13.12 \text{ ft}$$

$$V = \frac{Q}{Wh} = \frac{500000}{(10560)\,(13.12)} = 3.61 \text{ ft/s}$$

$$u_* = \sqrt{ghS_0} = \sqrt{(9.81)\,(13.12)\,(7.576 \times 10^{-5})} = 0.179 \text{ m/s}$$

$$t_t = \frac{W^2}{hu_*} = \frac{(10560)^2}{(13.12)\,(0.179)} = 47.5 \times 10^6 \text{ s}$$

$$\begin{aligned} L &= Vt_t = (3.61)\left(47.5 \times 10^6\right) = 171 \times 10^6 \text{ ft} \\ &= 32000 \text{ mi} \end{aligned}$$

Problem 10.6

The tabulation on page 203 gives the velocity distribution and the suspended sand concentration for the fraction passing a 0.105-mm sieve and retained on a 0.074-mm sieve on the Missouri River. Given the slope 0.00012, flow depth 7.8 ft, river width 800 ft, and water temperature 7°C:

(a) plot the velocity profile V versus $\log z$ and concentration $\log C$ versus $\log\left[(h-z)/z\right]$;

(b) compute from the graphs and given data the following:

u_* = shear velocity, V = mean velocity, κ = von Karman constant, f = Darcy-Weisbach friction factor, and Ro= Rouse number;

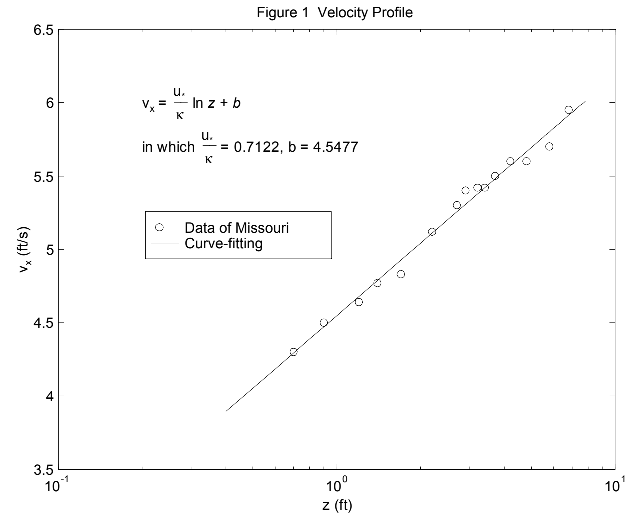
(c) compute the unit sediment discharge for this size fraction from field measurements; and

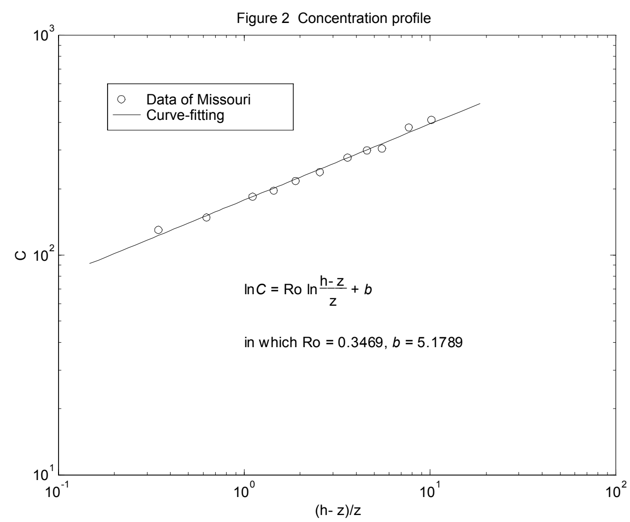
(d) calculate the flux-averaged concentration.

Solution Given:

$$\begin{aligned} d_s &= 0.074 - 0.105 \text{ mm}, \quad S_0 = 1.2 \times 10^{-4} \\ h &= 7.8 \text{ ft}, W = 800 \text{ ft}, T = 7°\text{C} \end{aligned}$$

(a) Velocity profile and concentration profile





The curve-fitting velocity profile can also be written

$$\begin{aligned} v_x &= \frac{u_*}{\kappa}\ln\frac{z}{h} + \frac{u_*}{\kappa}\ln h + b \\ &= \frac{u_*}{\kappa}\ln\xi + v_{x\,\max} \end{aligned}$$

in which $\xi = z/h$, and $v_{x\,\max} = \frac{u_*}{\kappa}\ln h + b = 0.7122\ln 7.8 + 4.5477 = 6.01$ ft/s.

The curve-fitting concentration profile can also read

$$C = e^b\left(\frac{1-\xi}{\xi}\right)^{\mathrm{Ro}}$$

(b) Computations of parameters

- Shear velocity

$$u_* = \sqrt{ghS_0} = \sqrt{(32.2)\,(7.8)\,(1.2\times 10^{-4})} = 0.173 \text{ ft/s}$$

- Mean velocity

  From Figure 1, the velocity profile reads

$$v_x = 0.7122 \ln z + 4.5477$$

  which gives the mean velocity as

$$\begin{aligned} V &= \frac{1}{7.8}\int_0^{7.8} (0.7122\ln z + 4.5477)\ dz = \frac{(4.5477)\,(7.8)}{7.8} + \frac{0.7122}{7.8}\int_0^{7.8} \ln z\, dz \\ &= \frac{35.47}{7.8} + \frac{0.7122}{7.8}\left(z\ln z - z\right)_0^{7.8} = \frac{35.47}{7.8} + \frac{0.7122}{7.8}\left(7.8\ln 7.8 - 7.8\right) \\ &= 5.30 \text{ ft/s} \end{aligned}$$

- von Karman constant

  From Figure 1, we get

$$\frac{u_*}{\kappa} = 0.7122$$

  which gives

$$\kappa = \frac{u_*}{0.7122} = \frac{0.173}{0.7122} = 0.244$$

- Darcy-Weisbach friction factor

$$\frac{V}{u_*} = \sqrt{\frac{8}{f}}$$

  which gives

$$f = 8\left(\frac{u_*}{V}\right)^2 = 8\left(\frac{0.173}{5.3}\right)^2 = 0.00852$$

- Rouse number

  From Figure 2, we get

$$\text{Ro } = 0.3469$$

(c) Unit sediment discharge

$$\begin{aligned} q_t &= \int_{k_s'/30}^{h} v_x C dz \\ &= h\int_{\frac{k_s'}{30h}}^{1}\left(\frac{u_*}{\kappa}\ln\xi + v_{x\,\max}\right)e^b\left(\frac{1-\xi}{\xi}\right)^{\text{Ro}} d\xi \\ &= he^b\left\{\frac{u_*}{\kappa}\int_{\frac{k_s'}{30h}}^{1}\left(\frac{1-\xi}{\xi}\right)^{\text{Ro}}\ln\xi d\xi + v_{x\,\max}\int_{\frac{k_s'}{30h}}^{1}\left(\frac{1-\xi}{\xi}\right)^{\text{Ro}} d\xi\right\} \\ &= he^b\left\{\frac{u_*}{\kappa}J_2 + v_{x\,\max}J_1\right\} \end{aligned}$$

in which J_1 and J_2 are calculated by ($\frac{d_s}{30h} = \frac{0.0895\times10^{-3}}{(30)(7.8\times0.3048)} = 1.25\times10^{-5} \to 0$)

$$J_1 = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^{\mathrm{Ro}} d\xi = \frac{\mathrm{Ro}\pi}{\sin(\mathrm{Ro}\pi)}$$

$$J_2 = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^{\mathrm{Ro}} \ln\xi d\xi = -J_1 f\,(\mathrm{Ro})$$

where

$$f\,(x) = 0.4228 - \ln(2-x) + \frac{1}{1-x} + \frac{1}{2\,(2-x)} + \frac{1}{24(2-x)^2}$$

Now,

$$\mathrm{Ro}\ = 0.3469$$

$$J_1 = \frac{0.3469\pi}{\sin\,(0.3469\pi)} = 1.229$$

$$\begin{aligned} f\,(0.3469) &= 0.4228 - \ln(2-0.3469) + \frac{1}{1-0.3469} + \frac{1}{2(2-0.3469)} + \frac{1}{24(2-0.3469)^2} \\ &= 0.4228 - 0.5027 + 1.5312 + 0.3025 + 0.01525 = 1.769 \end{aligned}$$

$$J_2 = -\,(1.229)\,(1.769) = -2.174$$

In addition, we have

$$h = 7.8 \text{ ft}, \quad b = 5.1789, \quad \frac{u_*}{\kappa} = 0.7122 \text{ ft/s} \quad v_{x\,\max} = 6.01 \text{ ft/s}$$

Thus,

$$\begin{aligned} q_t &= he^b\left\{\frac{u_*}{\kappa}J_2 + v_{x\,\max}J_1\right\} \\ &= (7.8)\,e^{5.1789}\,[(0.7122)\,(-2.174) + (6.01)\,(1.229)] \\ &= (1384.4)\,(-1.5483 + 7.8363) \\ &= 8082 \text{ ft } \cdot \text{ ft/s } \cdot \text{mg/l} \end{aligned}$$

(d) Flux-averaged concentration

$$\overline{C} = \frac{q_t}{hV} = \frac{8082 \text{ ft } \cdot \text{ ft/s } \cdot \text{mg/l}}{(7.8 \text{ ft})(5.30 \text{ ft/s})} = 195.5 \text{ mg/l}$$

i.e.

$$\boxed{\overline{C} = 195.5 \text{ mg/l}}$$

Appendix Program

```
%Problem 10.6 (page 202)

yv = [0.7 0.9 1.2 1.4 1.7 2.2 2.7 2.9 3.2 3.4...
     3.7 4.2 4.8 5.8 6.8];   %ft
v = [4.3 4.5 4.64 4.77 4.83 5.12 5.30 5.40...
     5.42 5.42 5.50 5.60 5.60 5.70 5.95]; %ft/s

yc = [0.7 0.9 1.2 1.4 1.7 2.2 2.7 3.2 3.7 4.8 5.8];   %ft
C = [411 380 305 299 277 238 217 196 184 148 130];  %mg/l

figure(1)
clf
semilogx(yv,v,'o'), hold on
cv = polyfit(log(yv),v,1);
y = 0.4:0.1:7.8;
fit = polyval(cv,log(y));
semilogx(y,fit)
title('Figure 1  Velocity Profile')
sxlabel('z (ft)')
sylabel('v_x (ft/s)')
stext(0.2,6,'v_x = \frac{u_*}{\kappa} ln {\i z} + {\i b}')
stext(0.2,5.7,'in which \frac{u_*}{\kappa} = 0.7122, b = 4.5477')
legend('Data of Missouri','Curve-fitting')
fixstext

figure(2)
clf
loglog((7.8-yc)./yc,C,'o'), hold on
cc = polyfit(log((7.8-yc)./yc),log(C),1);
y = 0.4:0.1:6.8;
fit = polyval(cc,log((7.8-y)./y));
loglog((7.8-y)./y,exp(fit))
title('Figure 2  Concentration profile')
sxlabel('(h-z)/z')
sylabel('C')
stext(1,70,'ln{\i C} = Ro ln\frac{h-z}{z} + {\i b}')
stext(1,40,'in which Ro = 0.3469, {\i b} = 5.1789')
legend('Data of Missouri','Curve-fitting')
fixstext
```

Problem 10.7

Bank erosion of fine silts occurs on a short reach of a 100-m-wide meandering river at a discharge of 750 m^3/s. If the riverbed slope is 50 cm/km and the flow depth is 5 m, and if the mass wasted is of the order of 10 metric tons per hour, determine the distance required for complete mixing in the river, the maximum concentration at that point, and the average sediment concentration.

Solution

$$W = 100 \text{ m}, \quad Q = 750 \text{ m}^3/\text{s}, \quad S = 0.0005, \quad h = 5 \text{ m}$$

$$V = \frac{Q}{Wh} = \frac{750}{(100)(5)} = 1.5 \text{ m/s}$$

$$\dot{M} = 10 \text{ tons/hour} = \frac{10\,(1000)\ \text{kg}}{3600 \text{ s}} = 2.78 \text{ kg/s}$$

$$u_* = \sqrt{ghS} = \sqrt{(9.81)(5)(0.0005)} = 0.1566 \text{ m/s}$$

$$\varepsilon_t = 0.15 h u_* = (0.6)(5)(0.1566) = 0.117 \text{ m}^2/\text{s}$$

$$L = 0.06 \frac{W^2}{\varepsilon_t} V = (0.06)\frac{100^2}{0.117}(1.5) = 7692 \text{ m}$$

$$C(x,y) = \frac{\dot{M}}{h\sqrt{4\pi\varepsilon_t V}} \exp\left(-\frac{y^2 V}{4\varepsilon_y x}\right)$$

$$C_{\max} = \frac{\dot{M}}{h\sqrt{4\pi\varepsilon_t V}} = \frac{2.78}{(5)\sqrt{4\pi(7692)(1.5)}} = 4.265 \text{ kg/m}^3 = 4265 \text{ mg/l}$$

Chapter 11

Total load

Problem 11.1

Compute the average sediment concentration C_{ppm} in an alluvial canal using the methods of Engelund and Hansen, Ackers and White, and Yang. The bed material (specific gravity 2.65) has the following particle size distribution:

Fraction diameter (mm)	Geometric mean (mm)	Fraction by weight (Δp_i)
0.062-0.125	0.088	0.04
0.125-0.25	0.177	0.23
0.25-0.5	0.354	0.37
0.5-1.0	0.707	0.27
1.0-2.0	1.414	0.09

The canal carries a discharge of 105 m^3/s with a water temperature of 15°C. The channel has a slope of 0.00027, an alluvial bed width of 46 m, a flow depth of 2.32 m, and a sideslope of 2:1.

Solution The results are Engelund-Hansen, 319 ppm; Ackers-White, 399 ppm; and Yang, 117 ppm. The equations can be found in pages 214-217. A MatLab program has been written as follows:

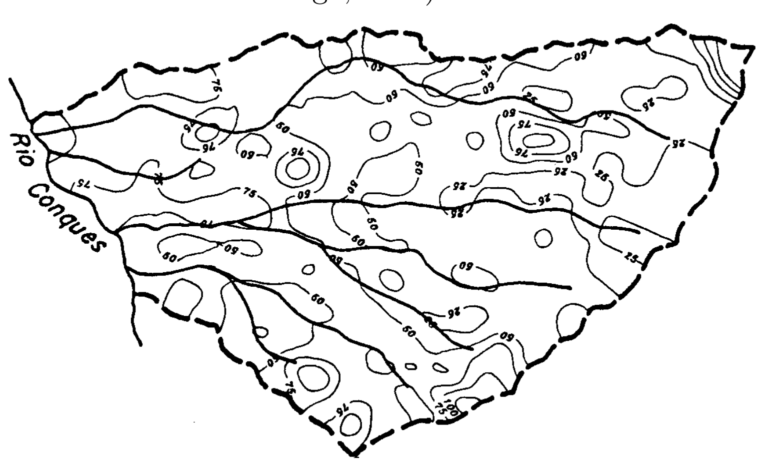
```
% Problem 11.1

Q = 105; % m^3/s
T = 15;  % C
S_f = 2.7e-4;
B = 46;  % m
h = 2.32; % m
SS = 2;
```

```
A = B*h + 2*h.^2;
P = B + 2*sqrt(5)*h;
R_h = A/B;
V = Q/A;

d_s = [0.088 0.177 0.354 0.707 1.414]; % mm
dp_i = [0.04 0.23 0.37 0.27 0.09];
d_s = d_s.*1e-3;

% Engelund-Hansen's method
C_w = hansen(R_h,S_f,V,d_s).*dp_i; sum1 = sum(C_w); C1_ppm = 1e6*sum1

% Ackers and White's method
C_w = ackers(T,d_s,u_star,V,h).*dp_i; sum2 = sum(C_w); C2_ppm = 1e6*sum2

% Yang's method
d_s = d_s*1000;
Cppm = yang(T,u_star,d_s,V,S).*dp_i; C3_ppm = sum(Cppm)

% Yang's method
function Cppm = yang(T,u_star,d_s,V,S)

nu = viscous(T);
w = rubey(T,d_s);
d_s = d_s.*1e-3;
Re = u_star.*d_s./nu;
if Re >= 70
 Vc = 2.05.*ones(size(d_s));
else
 Vc = 2.5./(log10(Re) - 0.06) + 0.66;
end

log_Cppm = 5.435 - 0.286.*log10(w.*d_s./nu) - 0.457.*log10(u_star./w)...
           + (1.799 - 0.409.*log10(w.*d_s./nu) - 0.314.*log10(u_star./w...
           )).*log10(V.*S./w - Vc.*S);

Cppm = 10.^log_Cppm;

%Engelund-Hansen method
function C_w = hansen(R_h,S_f,V,d_s)
% C_w = hansen(R_h,S_f,V,d_s)

C_w = 0.05.*(2.65./1.65).*V.*S_f./(1.65.*9.81.*d_s).^0.5.*(R_h.*...
```

```
      S_f./1.65./d_s).^0.5;

% Ackers-White's method
function C_w = ackers(T,d_s,u_star,V,h)

nu = viscous(T);
d_star = (1.65*9.81/nu.^2).^(1/3).*d_s;

if d_star >=60
 c_AW1 = 0; c_AW2 = 0.025; c_AW3 = 0.17; c_AW4 = 1.5;
else
 c_AW1 = 1 - 0.56.*log10(d_star);
 c_AW2 = 10.^(2.86.*log10(d_star) -log10(d_star).^2 - 3.53);
 c_AW3 = 0.23./sqrt(d_star) + 0.14;
 c_AW4 = 9.66./d_star + 1.34;
end

c_AW5 = u_star.^c_AW1./sqrt(1.65.*9.81.*d_s).*(V./sqrt(32)./...
        log10(10.*h./d_s)).^(1-c_AW1);

C_w = c_AW2.*2.65.*d_s./h.*(V./u_star).^c_AW1.*(c_AW5/c_AW3...
      - 1).^c_AW4;
```

Problem 11.2

The Conca de Tremp watershed covers 43.1 km^2 in Spain. The elevation ranges from 530 to 1,460 m above sea level; the climate is typically Mediterranean with 690 mm of mean annual precipitation and a 12.5°C mean annual temperature. The Mediterranean forest has been depleted and the region has been intensively farmed for centuries. With reference to the following upland erosion map in tons/hectare-year (after Julien and Gonzalez del Tanago, 1991):



(a) estimate the gross upland erosion and the sediment yield of the watershed;

(b) how does the erosion rate compare with (i) the geological erosion rate, 0.1 tons/acre-year (1 acre = 0.40468 hectare); (ii) accelerated erosion rates for pasture, 5 tons/acre-year; and (iii) the erosion rate of urban development, 50 tons/acre-year?

Solution (a) From the map, the average erosion rate is approximated 50 tons/hectare-year. The watershed covers 4310 hectares and the annual erosion loss is $50 \times 4310 = 215,500$ tons/yaer.

(b) The erosion rate of 50 tons/hectare-year corresponds to 20 tons per acre per yaer. This rate exceeds the geological erosion rate (0.1 ton/acre-year) and the accelerated erosion rate (5 tons/acre-year), but it is less than the erosion rate of urban development of 50 tons/acre-year.

Problem 11.3

Consider sediment transport in the Elkhorn River, Waterloo, Nebraska, given the total drainage area of 6,900 Mi^2 . The flow-duration curve and the sediment-rating curve are detailed in the following tabulations:

Flow-duration curve		Sediment-rating curve	
% time exceeded	Discharge (ft^3/s)	Discharge (ft^3/s)	Suspended load (tons/day)
0.05	37,000	280	250
0.30	15,000	500	600
1.00	9,000	800	1,000
3.25	4,500	1,150	3,000
10.00	2,100	1,800	8,000
20.00	1,200	2,300	18,000
30.00	880	4,200	40,000
40.00	710	6,400	90,000
50.00	600	8,000	300,000
60.00	510	10,000	500,000
70.00	425		
80.00	345		
90.00	260		
96.75	180		

Calculate (a) the mean annual suspended sediment load using the flow-duration /sediment-rating-curve method; and (b) the sediment yield per square mile.

Solution The sediment-rating curve is fitted to be

$$Q_s = 9.6 \times 10^{-4} Q^{2.139} \tag{11.1}$$

in which Q is in ft^3/s and Q_s in tons/day.

The time corresponding to each discharge in a year is

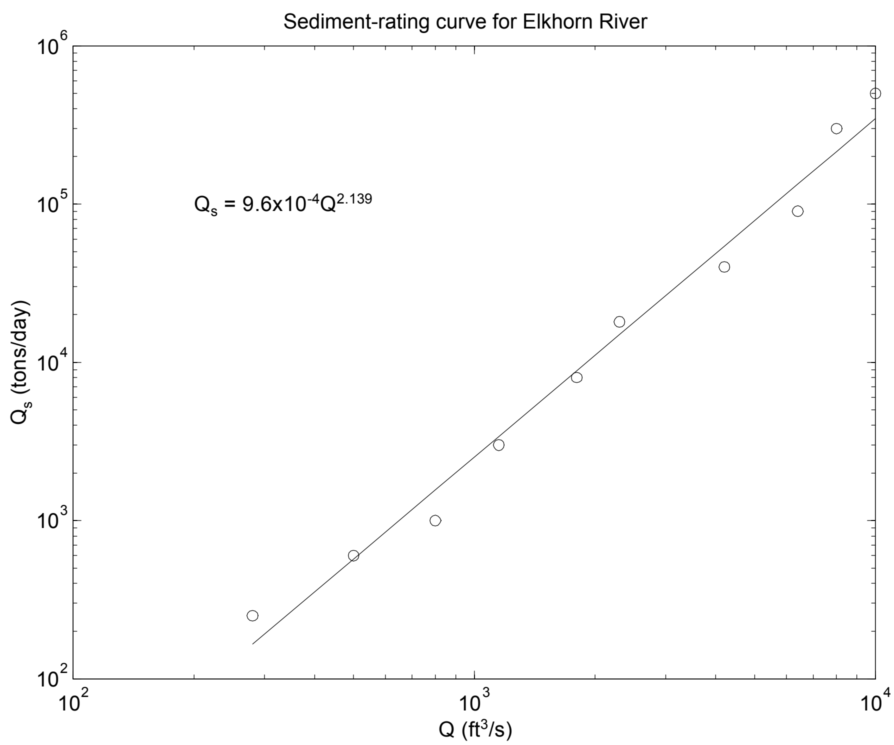
$$dt_i = 365\,(p_i - p_{i-1})\,, \quad \text{in days}$$

Given a discharge, we can find the corresponding sediment discharge Q_{si} from (11.1), then the annual sediment discharge is

$$Q_{sa} = \sum Q_{si} dt_i = 3.64 \times 10^6 \text{ tons/year}$$

The sediment yield per square mile is

$$Y_s = \frac{3.64 \times 10^6}{6900} = 527 \text{ tons/mi}^2\text{-year}$$



MatLab Program

```
% Problem 11.3, page 240

% Sediment-rating curve
Q = [280 500 800 1150 1800 2300 4200 6400 8000 10000];
Qs= [250 600 1000 3000 8000 18000 40000 90000 300000 500000];
loglog(Q, Qs, 'o'), hold on
```

```
c = polyfit(log(Q),log(Qs),1);
fit0 = polyval(c,log(Q));
fit = exp(fit0);
loglog(Q, fit)

sxlabel('Q (ft^3/s)')
sylabel('Q_s (tons/day)')
stext(200, 1e5, 'Q_s = 9.6x10^{-4}Q^{2.139}')
title('Sediment-rating curve for Elkhorn River')

% The mean annual suspended sediment load
p = [0.05 0.3 1.0 3.25 10 20 30 40 50 60 70 80 90 96.75];
Q = [37000 15000 9000 4500 2100 1200 880 710 600 510 425 345 260 180];

dp(1) = 0.05;
for i = 2:14, dp(i) = p(i) - p(i-1); end
dp = dp./100;

dt = dp*365;

Qs = 9.6e-4.*Q.^2.139;

Qs_a = sum(Qs.*dt)

% The sediment yield per square mile is
Qs_ps = Qs_a/6900
```

Chapter 12

Reservoir Sedimentation

Problem 12.1

From the data given in Case Study 12.2 on the Molineros Reservoir Project,

(a) determine the trap efficiency and the specific weight of sediment deposits after 10 years;

(b) use the flow-duration/sediment-rating-curve method to estimate the annual sediment load;

(c) calculate the life expectancy of the reservoir; and

(d) examine the impact of a 1,000-year flood in the next five years on the life expectancy of the reservoir.

Solution $L = 45,000\,\text{m}$, $V = 2.98\ \text{km}^2$, $Q_{ave} = 47.2\ \text{m}^3/\text{s}$, $\bar{h} = 100\ \text{m}$, $W = 600\,\text{m}$, $q_{ave} = Q_{ave}/W = 0.0787\,\text{m}^2/\text{s}$.

(a) Settling velocity:

	ω (mm/s)	$T_{Eave} = 1 - e^{-L\omega_i/q_{ave}}$
Sand	30	1
Silt	0.1	1
Clay	0.001	0.5

Dry specific weight:

	d_s (mm)	γ_{md1}	κ	γ_{md10}	$\Delta p_i \gamma_{md10}$
Sand	0.25	93	0	93	23.25
Silt	0.65	65	5.7	70.7	45.9
Clay	0.10	30	16	46	4.6

(b) Flow-duration sediment-rating-curve method. From Figure CS12.2.1, we have

Time intervals (%)	Δp	Q (m^3/s)	C (mg/l)	$Q_s = 0.0864QC$ (tons/day)	$Q_s\Delta p = 365Q_s\Delta p$ (metric tons/year)
0-2	0.02	1000	100,000	8.6×10^6	62.8×10^6
2-5	0.03	400	50,000	1.73×10^6	18.9×10^6
5-10	0.05	200	20,000	345,000	6.3×10^6
10-20	0.10	100	10,000	86,000	3.1×10^6
20-40	0.20	50	5,000	22,000	1.6×10^6
40-100	0.60	20	2,000	3,500	0.8×10^6
				$\sum$	93.5×10^6

(c) The life expectation of the reservoir is calculated from Eq. (12.12) to be

$$T_R = \frac{V\gamma_{mdr}}{\sum T_{E_i}\Delta p_i Q_{\tau i}} = \frac{3\times10^9\text{m}^3 1.18 \text{ tons-yaers}}{\text{m}^3\,(1\times100\times10^6)\text{ tons}} = 35.4 \text{ years}$$

(d) From Figure CS12.2.2, the 1000 year flood would provide a load about 8×10^6 tons. Given γ_{mdr} of 1.2 tons/m^3, this corresponds to a volume of $V = L/\gamma_{md} = 8\times10^6$ tons-m^3/1.2 tons= 6.7×10^6 m^3. Given the reservoir volume of 3×10^9 m^3, this corresponds to less than 1% of the storage capacity of this reservoir.