

TURBULENT SHEAR STRESS IN HETEROGENEOUS SEDIMENT-LADEN FLOWS

By Hyoseop Woo,¹ Associate Member, ASCE, and Pierre Y. Julien,² Member, ASCE

INTRODUCTION

Current knowledge of the mechanics of alluvial channels depends very largely on calculations of turbulent shear stresses; typical examples are the beginning of motion of sediment particles and sediment transport in alluvial channels. If shear stress can be well defined in clear-water flows, comparatively little is known about shear stresses in sediment-laden flows. Einstein and Chien (1955) proposed a turbulent shear stress relationship that does not depend on the fall velocity of uniform sediment particles. Opposite results were later proposed by Zagustin (1969) and Willis (1972).

This study briefly reviews existing relationships for turbulent shear stress in homogeneous sediment-laden flows, and proposes a derivation of turbulent shear stress in heterogeneous sediment-laden flows. The analysis of turbulent shear stress for steady-uniform flows carrying heterogeneous sediment mixtures provides additional insight on the role played by sediment size gradation and fall velocity by size fraction.

PREVIOUS DERIVATIONS

Einstein and Chien (1955) first derived the turbulent shear stress relationship for sediment-laden flows. Without providing further clarification they first assumed that

$$\tau = -\rho_f \overline{u'(1-c)v} - \rho_s \overline{u'c(v-\omega)} \dots\dots\dots (1)$$

in which the shear stress τ depends on three constants and three variables. The constants are the fall velocity of uniform sediment particles ω , the mass density of the fluid ρ_f , and the mass density of sediment ρ_s ; the variables are the instantaneous velocity components u and v , and the volumetric sediment concentration c . These variables fluctuate around their respective time-average values \bar{u} , \bar{v} , and \bar{c} by the amounts u' , v' , and c' . The time-average values of the Reynolds stress components are obtained from the Prandtl mixing theory as follows:

$$-\overline{u'v'} = \epsilon_m \frac{d\bar{u}}{dy} \dots\dots\dots (2)$$

¹Res. Fellow, Water Resour. Engrg. Div., Korea Inst. of Constr. Tech., Umyeondong 142, Socho-gu, Seoul, Korea 150-010.

²Assoc. Prof., Dept. of Civ. Engrg., Engrg. Res. Ctr., Colorado State Univ., Ft. Collins, CO 80523.

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TABLE 1. Turbulent Shear Stress in Sediment-Laden Flows

Source (1)	Formula (2)
Einstein and Chien (1955)	$\tau = \rho_f \epsilon_m d\bar{u}/dy(1 + A\bar{c})$
Zagustin (1969)	$\tau = \rho_f \epsilon_m d\bar{u}/dy(1 + A\bar{c}) - \rho_s \omega \bar{u} \bar{c}$
Willis (1972)	$\tau = \rho_f \epsilon_m d\bar{u}/dy(1 + A\bar{c}) + (\rho_s - \rho_f) \bar{u} [\epsilon_m d\bar{c}/dy + \omega \bar{c}]$

$$-\overline{u'(1-c)'} = \epsilon_f \frac{d(1-\bar{c})}{dx} = 0 \dots\dots\dots (3)$$

for streamwise uniform concentrations

$$-\overline{u'c'} = \epsilon_s \frac{d\bar{c}}{dx} = 0 \dots\dots\dots (4)$$

for streamwise uniform concentrations where ϵ_f , ϵ_m , and ϵ_s denote the diffusion coefficients for fluid, momentum, and sediment, respectively. Using Eqs. 3 and 4, Eq. 1 can be reduced to

$$\tau = -\rho_f \overline{u'v'} (1 + A\bar{c}) \dots\dots\dots (5)$$

where

$$A = \frac{\rho_s - \rho_f}{\rho_f} \dots\dots\dots (6)$$

Zagustin (1969) provided a slightly different derivation in order to explain the reduction of the von Karman constant in sediment-laden flows. His final relationship, given in Table 1, differs from Eq. 5 and involves the fall velocity ω . Based on Taylor series expansion, Willis (1972) derived a third relationship for turbulent shear stress in sediment-laden flows. This relationship given in Table 1 also depends on ω . Woo and Julien (1989) reviewed these formulations and indicated that all three derivations are questionable. For instance, Einstein and Chien used Eq. 1 without justification and erroneously neglected the correlation between c and v . Questionable steps in Zagustin's derivation include the constant value of the sediment concentration and the use of the velocity fluctuation v' instead of v in the vertical direction. Willis' equation would reduce to Einstein and Chien's relationship if ϵ_m were substituted by ϵ_s .

TURBULENT SHEAR STRESS FOR HETEROGENEOUS SEDIMENT-LADEN FLOWS

The following derivation by size fractions (Woo 1985) clarifies the role played by the fall velocity of sediment particles from a graded sediment mixture. Consider the steady and uniform motion of a heterogeneous sediment-laden flow in an open channel. With reference to Fig. 1, the fluxes of water F_f and sediment F_s passing respectively through a horizontal plane of area dA_f and dA_s parallel to the main flow direction are

$$F_f = \rho_f v dA_f \dots\dots\dots (7)$$

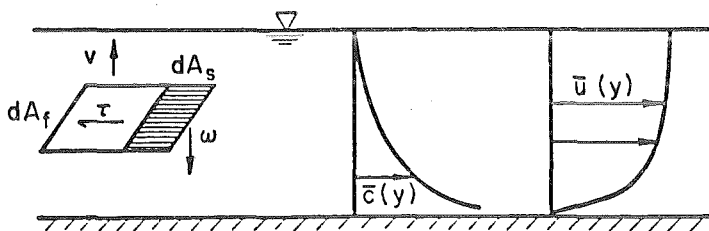


FIG. 1. Two-Dimensional Diagram for Sediment-Laden Flow

and

$$F_s = \sum_{i=1}^n \rho_{si}(v - \omega_i)dA_{si} \dots \dots \dots (8)$$

respectively. Here the subscript i denotes one of the n sediment size fractions and ω_i refers to the fall velocity of this size fraction.

The turbulent shear stress τ is thus obtained from adding the turbulent momentum fluxes in the x -direction for both the fluid and each fraction of sediments:

$$\tau = -u \left[\rho_f v \frac{dA_f}{dA} + \sum_{i=1}^n \rho_{si}(v - \omega_i) \frac{dA_{si}}{dA} \right] \dots \dots \dots (9)$$

It is assumed that the inertia of sediment particles is so small that the sediment velocity in the x -direction equals the fluid velocity, u , whereas in the y -direction the sediment velocity differs from the fluid velocity, v , by the settling velocity ω_i of the size fraction i . The following identities are defined:

$$\frac{dA_{si}}{dA} = c_i \dots \dots \dots (10)$$

$$c = \sum_{i=1}^n c_i \dots \dots \dots (11)$$

$$\frac{dA_f}{dA} = \frac{dA}{dA} - \frac{1}{dA} \sum_{i=1}^n dA_{si} = 1 - c \dots \dots \dots (12)$$

$$\rho_m = \rho_f + (\rho_s - \rho_f)c \dots \dots \dots (13)$$

With these four relationships (Eqs. 10–13), and assuming that ρ_s is constant for all size fractions ($\rho_{si} = \rho_s$), Eq. 9 reduces to

$$\tau = -\rho_m uv + \rho_s \sum_{i=1}^n \omega_i c_i u \dots \dots \dots (14)$$

Instantaneous values of the sediment concentration and the mass density of the mixture are defined as follows:

$$c = \bar{c} + c' = \sum_{i=1}^n \bar{c}_i + \sum_{i=1}^n c'_i \dots \dots \dots (15a)$$

$$\rho_m = \bar{\rho}_m + \rho'_m \dots \dots \dots (15b)$$

$$\bar{\rho}_m = \rho_f + (\rho_s - \rho_f)\bar{c} \dots \dots \dots (15c)$$

$$\rho'_m = (\rho_s - \rho_f)c' \dots \dots \dots (15d)$$

in which the overbar denotes the time-average value of the parameter and the prime denotes fluctuations of the same parameter. Substitution of Eqs. 15a and 15d into Eq. 14 and the application of the Reynolds averaging process gives

$$\tau = -(\bar{\rho}_m + \rho'_m)(\bar{v} + v')(\bar{u} + u') + \rho_s \sum_{i=1}^n \omega_i (\bar{c}_i + c'_i)(\bar{u} + u') \dots \dots \dots (16)$$

The expression for τ is the time-average value of the shear stress. Under equilibrium conditions, the time-average mass transfer of water in the y-direction can be written as

$$\left(1 - \sum_{i=1}^n c_i\right)v = 0 \dots \dots \dots (17a)$$

which can be rewritten as

$$\sum_{i=1}^n c'_i v' = \bar{v}(1 - \bar{c}) \dots \dots \dots (17b)$$

For sediments, the time-average mass transfer in the y-direction is given by

$$\sum_{i=1}^n c_i(v - \omega_i) = 0 \dots \dots \dots (18a)$$

i.e.

$$\bar{v} = \sum_{i=1}^n \omega_i \bar{c}_i \neq 0 \dots \dots \dots (18b)$$

This relationship (Eq. 18b) demonstrates that the average velocity \bar{v} in the vertical direction is not zero but proportional to the fall velocity of sediment particles and the concentration of each size fraction.

With the aid of Eqs. 3, 4, 17, and 18, the shear stress relationship (Eq. 16) can be expanded (see Appendix I) and simplified to

$$\tau = -\overline{\rho_m u' v'} \dots \dots \dots (19)$$

This relationship is valid without any restriction on the triple correlation. Further simplification of Eq. 19 is possible when neglecting the triple correlation term $\overline{\rho_m u' v'}$ can be justified. This is possible from an order of magnitude analysis of the terms in Eq. 15b. Indeed, when the mass density fluctuation ρ'_m is negligible compared to $\bar{\rho}_m$, Eq. 19 further reduces to

$$\tau = -\rho_f \overline{u' v'} (1 + A\bar{c}) \dots \dots \dots (20)$$

which is identical to the relationship derived by Einstein and Chien. This

derivation shows that turbulent shear stress in sediment-laden flows is similar to that of the water flow except that the fluid mass density ρ_f is replaced with the density of the water sediment mixture ρ_m .

Eq. 19 is the chief result of this analysis, which demonstrates that the sediment size distribution and the fall velocity of sediment particles do not affect turbulent shear stress of heterogeneous sediment mixtures under steady-uniform flow conditions.

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APPENDIX I. DERIVATION

Applying the Reynolds averaging method to Eq. 16 yields

$$\begin{aligned}
 -\tau &= \bar{\rho}_m \bar{u}\bar{v} + \bar{\rho}_m \overline{u'v'} + \bar{v} \overline{\rho'_m u'} + \bar{u} \overline{\rho'_m v'} + \overline{\rho'_m u'v'} \\
 &- \rho_s \bar{u} \sum_{i=1}^n \omega_i \bar{c}_i - \rho_s \sum_{i=1}^n \omega_i \overline{c'_i u'} \dots\dots\dots (21)
 \end{aligned}$$

The last term of Eq. 21 can be dropped due to Eq. 4 because $d\bar{c}/dx = 0$ under steady-uniform flow conditions. Note that Eq. 4, which is critical to this derivation, is also substantiated by the findings of Fischer et al. (1979), who stated that "the total mass transport in the streamwise direction is proportional to the concentration gradient in the streamwise direction." With Eq. 18b, the remaining summation reduces to $-\rho_s \bar{u}\bar{v}$. Now expanding ρ'_m from Eq. 15d gives

$$-\tau = \bar{\rho}_m \bar{u}\bar{v} + \overline{\rho'_m u'v'} + \bar{v} \overline{u'(\rho_s - \rho_f)c'} + \bar{u} \overline{v'(\rho_s - \rho_f)c'} - \rho_s \bar{u}\bar{v} \dots\dots\dots (22)$$

The third term on the right-hand side vanishes due to Eq. 4 and the following term reduces to $\bar{u}\bar{v}(1 - \bar{c})(\rho_s - \rho_f)$ according to Eq. 17b. Expanding $\bar{\rho}_m$ from Eq. 15c yields the final form of Eq. 22

$$\begin{aligned}
 -\tau &= \rho_f \bar{u}\bar{v} + \rho_s \bar{u}\bar{v}\bar{c} - \rho_f \bar{u}\bar{v}\bar{c} + \overline{\rho'_m u'v'} + \rho_s \bar{u}\bar{v} \\
 &- \rho_s \bar{u}\bar{v}\bar{c} - \rho_f \bar{u}\bar{v} + \rho_f \bar{u}\bar{v}\bar{c} - \rho_s \bar{u}\bar{v} \dots\dots\dots (23)
 \end{aligned}$$

which simply reduces to

$$\tau = -\overline{\rho'_m u'v'} \dots\dots\dots (24)$$

APPENDIX II. REFERENCES

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APPENDIX III. NOTATION

The following symbols are used in this paper:

- A = density quotient; $A = (\rho_s - \rho_f)/\rho_f$;
- c = volumetric sediment concentration;
- dA = area of elementary plane parallel to main flow direction;
- dA_s = portion of dA occupied by sediment;
- dA_f = portion of dA occupied by water;
- F_f = mass flux of fluid;
- F_s = mass flux of sediment;
- n = number of sediment size fractions;
- u = longitudinal fluid velocity;
- v = vertical fluid velocity;
- x = longitudinal distance in downstream direction;
- y = vertical distance from channel bed;
- ϵ_f = diffusion coefficient for fluid mass (water);
- ϵ_m = diffusion coefficient for fluid momentum;
- ϵ_s = diffusion coefficient for sediment;
- ρ_f = density of fluid (water);
- ρ_m = density of sediment-water mixture;
- ρ_s = density of sediment;
- τ = shear stress; and
- ω = settling velocity of sediment in flow.

Subscripts and Superscripts

- ' = fluctuating value of parameter;
- = time-average value of parameter; and
- i = size fraction of sediment.