

SUSPENSION OF LARGE CONCENTRATIONS OF SANDS

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ABSTRACT: Existing methods based on the diffusion theory have been extended to determine the sediment concentration distribution for large concentrations of sand. The modifications for volumetric concentrations ranging from 4 to 20% include decreased fall velocity, and increased viscosity and specific weight of the suspension. The sediment concentration profiles are sensitive to the velocity profiles and the fall velocity of sediment particles. In turbulent flows, either a power law or a two-layered logarithmic law can be used to describe the velocity profiles and the sediment diffusivity. The proposed method accurately depicts the deviation from the Rouse equation at large concentrations of sands, and the agreement with sediment concentration profiles measured by Einstein and Chien is excellent. The proposed numerical solution gradually reduces to the relationships of Karim and Kennedy, and Rouse as the volumetric concentration decreases below four percent.

INTRODUCTION

Flowing water has the ability to suspend large quantities of sand particles depending on the availability of sediment and the transport capacity of the flow. To describe equilibrium conditions in turbulent flows, Rouse (1937) used Schmidt's (1925) diffusion equation and derived the following equation for the vertical distribution of sediment particles

$$\frac{C}{C_a} = \left[\left(\frac{d-y}{y} \right) \left(\frac{a}{d-a} \right) \right]^Z \dots\dots\dots (1)$$

in which

$$Z = \frac{W_s}{\beta \kappa u_*} \dots\dots\dots (2)$$

The sediment concentration C at a distance y above the bed depends on the total depth d and the reference concentration C_a at the reference height a . The exponent Z expresses the ratio of the fall velocity W_s of sediment particles to the product $\beta \kappa u_*$ involving the shear velocity u_* , the von Kármán constant κ , and the ratio β of the sediment diffusivity to the fluid momentum diffusivity. For sediment-laden flows with concentrations below a few percent, Rouse's relationship (Eq. 1) has been verified extensively with laboratory and field measurements. Einstein and Chien (1955) experimentally found that in highly concentrated flows, however, the distribution of sands in suspension deviates progressively from what is

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predicted by the Rouse equation. Sediment concentration becomes more uniform as the local sediment concentration exceeds a value of 100 g/l ($C \cong 3.8\%$ by volume).

Chien and Wan (1965) reanalyzed the experimental results of Einstein and Chien for the velocity and sediment concentration profiles. After considering the fall velocity reduction according to Richardson and Zaki's (1954) formula, a modified power form of the velocity profile was used to provide a graphical solution to the Schmidt diffusion equation. More recent contributions by Ordonez (1970), Lavelle and Thacker (1978), and van Rijn (1984) are based on similar concepts with Richardson and Zaki's equation for the fall velocity reduction and Schmidt's formulation of the diffusion equation.

This study examines the vertical distribution of sands in suspension at volumetric concentrations ranging between 4 and 20%. At such high concentrations, Schmidt's approximation of the diffusion equation no longer applies. The properties of the fluid mixture in terms of viscosity and density of the suspension are modified accordingly. Theoretical sediment concentration profiles are determined and compared with Einstein and Chien's data, which is a unique data set at such large concentration of sands in suspension.

DIFFUSION OF LARGE CONCENTRATIONS OF SAND PARTICLES

After considering the diffusion equations proposed by Schmidt (1925), Halbronn (1949), and Hunt (1954), the relationship best suited to large concentration of sands involves the diffusion coefficient of sand particles ϵ_s and the representative fall velocity W_p as follows

$$\epsilon_s \frac{dC}{dy} + C(1 - C)W_p = 0 \quad \dots\dots\dots (3)$$

This equation derived by Hunt improves upon Schmidt's formulation ($W_s C + \epsilon_s dC/dy = 0$) for volumetric concentration between 4 and 20% because it satisfies continuity of sediment and water.

The representative fall velocity of the mixture W_p varies with the volumetric concentration C as defined by Richardson and Zaki (1954) or Maude and Whitmore (1958) as

$$W_p = \omega_p(1 - C)^\alpha \quad \dots\dots\dots (4)$$

Although there is no strict rule guiding the selection of the exponent α , it varies with the particle Reynolds number and the particle shape, and decreases from 4.65 to 2.35 as the size of noncohesive particles increases from silts to gravels. The representative fall velocity in clear water ω_p is calculated from the fall velocity ω_i of individual size fractions as:

$$\omega_p = \sum_{i=1}^N \frac{\omega_i \bar{C}_i}{\bar{C}} \quad \dots\dots\dots (5)$$

The subscript i denotes each one of the N -size fractions of sands in suspension, whereas the overbar represents depth-averaged values of the concentration.

The diffusion coefficient ϵ_s in Eq. 3 is determined from shear stress analysis. At large concentrations of sands, the total shear stress τ is balanced by the sum of the viscous shear stress τ_v and the turbulent shear stress τ_t .

$$\tau = \tau_t + \tau_v \quad \dots \dots \dots (6)$$

The reason for adding the viscous shear stress to the commonly used turbulent shear stress is to describe the variability of τ_v due to the increase in sediment concentration. Of course this added term τ_v can be neglected as the concentration becomes vanishingly small in turbulent flows.

The turbulent shear stress relationship derived by Einstein and Chien (1955) considers the turbulent momentum exchange in fluid-particle mixtures

$$\tau_t = \rho_f \epsilon_m \frac{d\bar{u}}{dy} (1 + AC) \quad \dots \dots \dots (7)$$

In clear water ($C = 0$), this equation reduces to Boussinesq's formulation given by the product of the fluid density ρ_f with the fluid momentum diffusivity ϵ_m and the gradient of the time-averaged velocity \bar{u} . The correction factor $A = (\rho_s - \rho_f)/\rho_f$ is a function of the densities of sand particles ρ_s and of the fluid ρ_f .

The viscous shear stress, τ_v , depends on the concentration of sand particles, the dynamic viscosity η of the fluid, and the velocity gradient

$$\tau_v = g(C)\eta \frac{d\bar{u}}{dy} \quad \dots \dots \dots (8)$$

The function $g(C) = 1 + 2.5 C + 10.05 C^2 + 0.00273 \exp(16.6 C)$ has been suggested by Thomas (1965) and reduces to Einstein's equation [$g(C) = 1 + 2.5 C$] at lower concentrations.

The total shear stress component τ depends on the unit weight of the suspension γ_m and the energy gradient S_e of the flow

$$\tau = \int_y^d \gamma_m S_e dy \quad \dots \dots \dots (9)$$

The bed shear stress is $\tau_0 = \rho_f u_*^2 (1 + A\bar{C})$. Accordingly, the vertical distribution of shear stress τ/τ_0 can be written as a function of the dimensionless depth $\zeta = y/d$

$$\frac{\tau}{\tau_0} = \frac{1 - \zeta + A \int_{\zeta}^1 C d\zeta}{1 + A\bar{C}} \quad \dots \dots \dots (10)$$

After substituting Eqs. 7, 8, and 10 into Eq. 6, the diffusion coefficient for suspended sands, ϵ_s , is obtained from the fluid momentum diffusivity coefficient ϵ_m according to the Reynolds analogy ($\epsilon_s = \beta \epsilon_m$)

$$\varepsilon_s = \beta \left[\frac{u_*^2 \left(1 - \zeta + A \int_{\zeta}^1 C \, d\zeta \right)}{\frac{d\bar{u}}{dy} (1 + AC)} - \frac{v_f g(C)}{(1 + AC)} \right] \dots \dots \dots (11)$$

in which v_f is the kinematic viscosity of the fluid ($\eta = \rho_f v_f$). The sediment diffusivity coefficient ε_s can be evaluated from Eq. 11 after the velocity gradient is determined.

Velocity Profiles

Similarities between velocity profiles for clear water and sediment-laden flows suggest the use of the logarithmic velocity profile. The velocity gradient can thus be expressed as

$$\frac{d\bar{u}}{dy} = \frac{u_*}{\kappa y} \dots \dots \dots (12)$$

Alternatively, the time-averaged velocity \bar{u} can be written as a power function of the dimensionless depth ζ and the depth-averaged velocity V

$$\frac{\bar{u}}{V} = n_1 \zeta^{n_2} \dots \dots \dots (13)$$

The empirical coefficients n_1 and n_2 depend on the flow characteristics. One can demonstrate that $n_1 = 1 + n_2$ in a two-dimensional flow, and the following analytical relationship between κ and n_2 , ($n_2 = u_*/\kappa V$), was derived by Zimmermann and Kennedy (1978) for open-channel flow. Note that Eq. 13 is strictly valid for a power formulation of velocity profiles.

VERTICAL DISTRIBUTION OF LARGE CONCENTRATIONS OF SANDS

The vertical distribution of sediments in suspension is obtained after substituting the turbulent diffusion coefficient (Eq. 11) and the representative fall velocity (Eq. 4) into the diffusion relationship (Eq. 3).

$$\frac{dC}{dy} = \frac{\omega_p}{\beta} \left[\frac{\frac{d\bar{u}}{dy} f(C)}{v_f g(C) \frac{d\bar{u}}{dy} - u_*^2 \left(1 - \zeta + A \int_{\zeta}^1 C \, d\zeta \right)} \right] \dots \dots \dots (14)$$

in which $f(C) = C(1 - C)^{\alpha+1}(1 + AC)$. Eq. 14 is of general applicability and can be solved, provided the velocity profile is accurately depicted. Substituting the velocity gradient from the power-law (first derivative of the velocity profile given by Eq. 13) into Eq. 14 yields

$$\frac{dC}{d\zeta} = \frac{f(C)Z_k}{g(C)H_k - \zeta^{1-n_2} \left(1 - \zeta + A \int_{\zeta}^1 C \, d\zeta \right)} \dots \dots \dots (15)$$

Two dimensionless parameters are defined: $Z_k = n_1 n_2 V \omega_p / \beta u_*^2$; and $H_k = n_1 n_2 v_f V / u_*^2 d$. From Eq. 15, concentration decreases with the relative depth

only when the denominator of the right-hand side of the equation is negative, which invalidates its applicability at the bed or near the surface.

Eq. 15 constitutes an integro-differential equation also called “Volterra Integral,” which after differentiating with respect to ζ transforms into the following second-order nonlinear ordinary differential equation

$$\frac{d^2C}{d\zeta^2} = \left(\frac{n_2-1}{\zeta}\right) \frac{dC}{d\zeta} - h(C) \left(\frac{dC}{d\zeta}\right)^2 - \left\{ \left[\frac{H_k}{Z_k}\right] \left[\frac{g'(C)}{f(C)}\right] \right\} \left(\frac{dC}{d\zeta}\right)^3 \dots\dots\dots (16)$$

in which

$$h(C) = \left\{ \frac{H_k}{Z_k} \left(\frac{n_2-1}{\zeta}\right) \frac{g(C)}{f(C)} + \frac{1}{Z_k} \left[\frac{1+AC}{f(C)}\right] \zeta^{1-n_2} - \frac{f'(C)}{f(C)} \right\} \dots\dots\dots (17)$$

The functions $f'(C)$ and $g'(C)$ denote derivatives of the functions $f(C)$ and $g(C)$ with respect to C . The fifth- and sixth-order Runge-Kutta method detailed in Woo (1985) provides a numerical solution to Eq. 16 after two boundary conditions are satisfied. First, $C = C_a$ at $\zeta = \zeta_a$ taking ζ_a as the dimensionless reference height. The second boundary condition of $\zeta = \zeta_a$ can be obtained from Eq. 15 as

$$\frac{dC}{d\zeta} = \frac{Z_k f(C_a)}{g(C_a)H_k - \zeta_a^{1-n_2}(1-\zeta_a)(1+A\bar{C})} \dots\dots\dots (18)$$

An iterative procedure has been used to solve Eq. 18. The first value of \bar{C} is obtained from Rouse’s equation. Few iterations are sufficient, since the solution to Eq. 18 is not sensitive to \bar{C} .

Logarithmic Velocity Profile

When using logarithmic velocity profiles in turbulent flows, the velocity gradient from Eq. 12 is substituted into Eq. 14. The solution to Eq. 14 is then very similar to the formulation presented with Eqs. 16, 17, and 18, except that: (1) $n_2 = 0$; (2) H_k is negligible; and (3) Z_k is replaced by $Z = \omega_p/\beta\kappa u_*$. Note that this result has not been obtained from Eq. 13 with $n_2 = 0$. Since n_2 does not appear in Eq. 12, it must therefore not appear in the resulting formulation of Eqs. 16, 17, and 18 for logarithmic velocity profiles; hence, $n_2 = 0$.

Approximate Suspended Sediment Distributions at Low Concentrations

Approximate analytical solutions to Eq. 14 are available for low sediment concentrations because several terms in Eqs. 14 and 15 can be simplified or neglected: (1) $f(C) = C(1 - C)^{\alpha+1} (1 + AC) \approx C$; (2) $A \int_{\zeta}^1 C d\zeta \approx 0$; and (3) $H_k \approx 0$. When τ_v is negligible compared to τ_t in Eq. 6, Eq. 15 then reduces to

$$\left(\frac{1}{\bar{C}}\right) \left(\frac{dC}{d\zeta}\right) + \left[\frac{Z_k}{(1-\zeta)\zeta^{1-n_2}}\right] = 0 \dots\dots\dots (19)$$

A graphical solution to this equation was reported by Laursen and Lin (1952), while Karim and Kennedy (1983) suggested the following series expansion formulation

$$\frac{C}{C_a} = \exp Z_k \left(\sum_{j=0}^{\infty} \frac{\xi_a^{j+n_2} - \xi^{j+n_2}}{j+n_2} \right) \dots\dots\dots (20)$$

When logarithmic velocity profiles are considered (with $n_2 = 0$, and Z_k replaced with Z), Eq. 19 simplifies further to

$$\left(\frac{1}{C} \right) \left(\frac{dC}{d\xi} \right) + \frac{Z}{\xi(1-\xi)} = 0 \dots\dots\dots (21)$$

The solution of Eq. 21 gives the well-known Rouse relationship (Eq. 1).

EXPERIMENTAL VERIFICATION OF THE SEDIMENT CONCENTRATION PROFILES

The theoretical profiles from Eq. 16 have been tested against one of the most comprehensive data sets on velocity and sediment concentration profiles for suspensions of large quantities of sand particles. From a total of 16 runs, the flume data collected by Einstein and Chien (1955) includes seven runs with local volumetric sediment concentrations between 4% and 23%. The flow and sediment characteristics of these runs are summarized in Table 1.

The velocity profiles of the hyperconcentrated flows are shown on both logarithmic (Fig. 1) and semilogarithmic scales (Fig. 2). The velocity measurements above the lowest point of the concentration measurement are used to calculate the slopes of the straight lines in these two figures. The values of n_2 are obtained from Fig. 1, and a two-layer model has been chosen to better describe the logarithmic velocity profiles in Fig. 2. The break in slope is observed near the lower 10% of the flow depth ($\xi \approx 0.1$). The slopes of the lower and the upper region define two values of the von Kármán coefficient (e.g., for run S-15, $\kappa_l = 0.241$ and $\kappa_u = 0.168$). The corresponding values of the Z factor of the Rouse equation are denoted Z_l and Z_u , respectively and the values are compiled in Table 1.

Figs. 3 and 4 illustrate the comparison between theoretical and measured sediment concentration profiles with the reference concentrations, C_a , taken at the lowest elevation near the bed. Excellent agreement is shown between the solution of Eq. 16 and the measurements of Einstein and Chien. In spite of the breakdown in two layers, both Rouse's linear profiles and Karim and Kennedy's slightly curved profiles remain far from the

TABLE 1. Hyperconcentrated Flow and Sediment Characteristics from Einstein and Chien Data

Run (1)	D (mm) (2)	κ_l (3)	κ_u (4)	n_2 (5)	C_a (6)	α (7)	β (8)	Z_k (9)	Z_l (10)	Z_u (11)
S-5	1.3	0.173	0.173	0.445	0.123	2.7	3.0	14.18	6.01	6.01
S-10	0.99	0.248	0.248	0.311	0.081	3.0	1.7	8.32	4.02	4.02
S-12	0.27	0.330	0.274	0.257	0.077	3.5	1.0	2.80	1.30	1.57
S-13	0.27	0.268	0.237	0.273	0.131	3.5	1.0	3.20	1.61	1.82
S-14	0.27	0.310	0.218	0.300	0.144	3.5	1.0	3.05	1.21	1.72
S-15	0.27	0.241	0.168	0.370	0.232	3.5	1.0	4.21	1.59	2.29
S-16	0.27	0.360	0.182	0.387	0.230	3.5	1.0	3.75	1.01	2.00

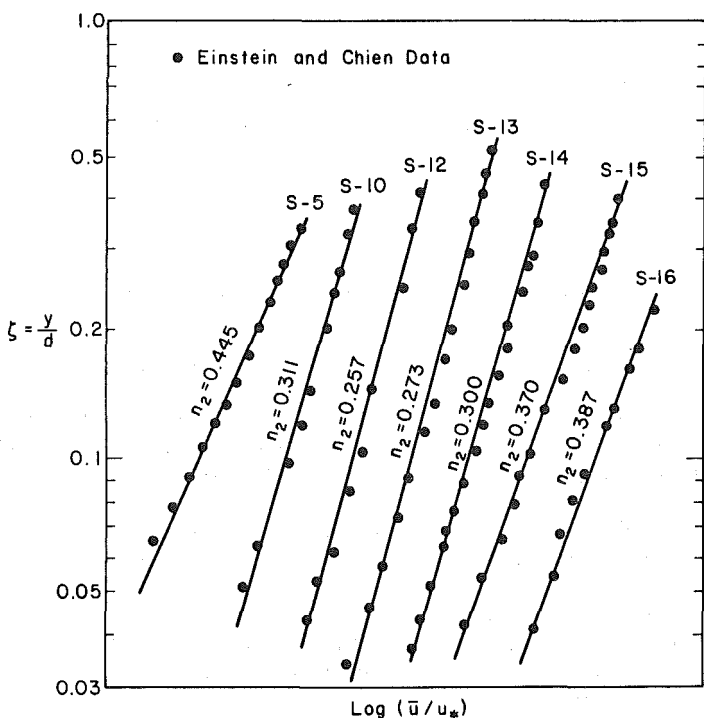


FIG. 1. Velocity Profiles at Large Concentrations of Sands

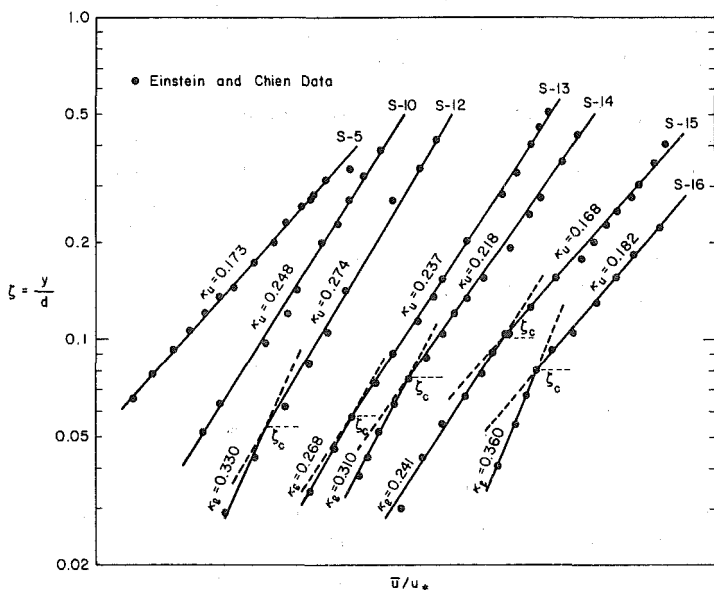


FIG. 2. Two-Layered Logarithmic Velocity Profiles at Large Concentration of Sands

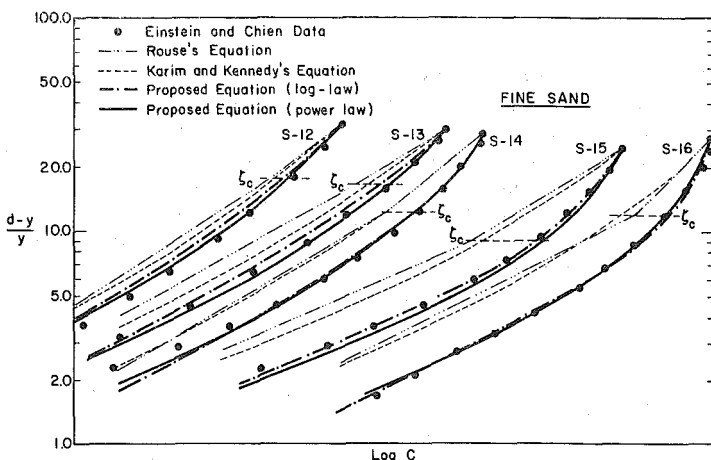


FIG. 3. Concentration Profiles at Large Concentration of Fine Sands

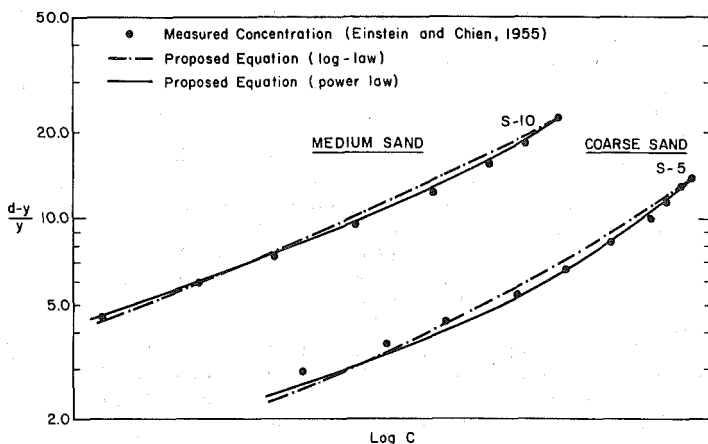


FIG. 4. Concentration Profiles at Large Concentrations of Medium and Coarse Sands

observed profiles. These relationships are best suited to sediment concentrations of less than about four percent volume.

SUMMARY AND CONCLUSIONS

The applicability of the diffusion approach to determine the vertical distribution of large concentrations of sand particles has been closely examined for concentrations of sands ranging from 4 to 20% by volume. Existing methods have been modified in the following aspects: (1) Hunt's diffusion equation (Eq. 3) is preferred to Schmidt's formulation because it satisfies continuity of fluid and sediments; (2) the fall velocity of sand particles decreases with sediment concentration; and (3) the derivation

includes the effects of increased viscosity and specific weight of the suspension.

The proposed theoretical profiles obtained from solving Eq. 16 depend on the velocity profiles. Solutions using both a power function or a two-layered logarithmic velocity profile demonstrate excellent agreement with the sediment concentration profiles measured by Einstein and Chien. At lower concentrations, Eq. 16 reduces gradually to the equations of Karim and Kennedy, and Rouse. The methodology thus presented extends the applicability of existing methods based on the diffusion equation to large concentrations (up to 20% by volume) of noncohesive particles in suspension. This methodology, however, is not expected to be applicable when substantial fractions of fine cohesive particles are added to the suspension.

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APPENDIX II. NOTATION

The following symbols are used in this paper:

- A = density quotient; $A = (\rho_s - \rho_f)/\rho_f$;
- a = reference height above the bed;
- C = volumetric sediment concentration;
- \bar{C} = depth-averaged sediment concentration;
- C_a = reference concentration;
- D = median particle size;
- d = channel depth;
- $f(C)$ = a function of sediment concentration;
- $g(C)$ = a function related to relative viscosity of cohesionless particle-fluid system;
- H_k = H -factor for the power-law equation;
- $h(C)$ = a coefficient for the differential equation (Eq. 17);
- i = one of the N -size fraction of sands;
- j = summation index in Karim and Kennedy's equation;
- N = number of size fractions;
- n_1 = coefficient of power-law equation;
- n_2 = exponent of power-law equation;
- S_e = energy gradient of channel;
- \bar{u} = time-averaged local velocity;
- u_* = shear velocity; $u_* = \sqrt{g d S_e}$;
- V = depth-averaged velocity;
- W_p = representative fall velocity reduced because of the concentration of sands (Eq. 4);
- W_s = fall velocity of sands in suspension (low concentration);
- y = vertical distance from bed;
- Z = exponent in the Rouse equation (Eq. 2);
- Z_k = Z -factor for the Karim and Kennedy relationship;
- Z_l = Z -factor in the lower portion of the velocity profile;
- Z_u = Z -factor in the upper portion of the velocity profile;
- α = exponent of the fall velocity reduction equation;
- β = ratio of sediment diffusivity, ϵ_s , to fluid momentum diffusivity, ϵ_m ;
- γ_m = unit weight of water-sediment mixture;
- ϵ_m = diffusion coefficient for fluid momentum;
- ϵ_s = diffusion coefficient for sands;
- ζ = dimensionless height; $\zeta = y/d$;
- ζ_a = dimensionless reference height; $\zeta_a = a/d$;
- η = dynamic viscosity of fluid;
- κ = von Kármán constant;
- κ_l = von Kármán constant for lower flow region;
- κ_u = von Kármán constant for upper flow region;
- ν_f = kinematic viscosity of fluid;
- ρ_f = density of fluid;
- ρ_s = density of sand particles;

- τ = shear stress;
- τ_0 = bed shear stress;
- τ_t = turbulent shear stress;
- τ_v = viscous shear stress;
- ω_i = fall velocity in clear water of a sand particle of the i -th size fraction; and
- ω_p = representative fall velocity in clear water (Eq. 5).