Efficient Algorithm for Computing Einstein Integrals

Junke Guo[1] and Pierre Y. Julien[2]

Abstract: Analytical approximations to Einstein integrals are proposed. The approximations represented by two fast-converging series are valid for all values of their arguments. Accordingly, the algorithm can be easily incorporated into professional software like *HEC-RAS* or *HEC-6* with minimum computational effort.

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Introduction

The Einstein bed load function is a landmark of modern sediment transport mechanics. It provides the first theoretical framework for sediment transport calculation, which guided many of the following researchers. Nevertheless, the computation of Einstein bed load function requires an estimation of two integrals J_1 and J_2, which cannot be integrated in closed form for most cases and are very slowly convergent for direct numerical integration because of singularity of the integrands near the bed (Nakato 1984). Einstein (1950) provided a numerical table and graphs to facilitate the calculation. Some mathematical software, such as *MatLab* and *Maple* can also be used to integrate them numerically. However, both methods cannot be easily implemented in professional software. For example, the widely used *HEC-RAS* and *HEC-6* do not include Einstein bed load function (U.S. Army Corps of Engineers 1993, 2003) probably because of the complexity. The purpose of this article is to provide a fast-converging algorithm to estimate Einstein integrals J_1 and J_2.

Einstein Integrals

In his bed load function, Einstein (1950) defined

$$J_1(z) = \int_E^1 \left(\frac{1-\xi}{\xi}\right)^z d\xi \qquad (1)$$

and

$$J_2(z) = \int_E^1 \left(\frac{1-\xi}{\xi}\right)^z \ln \xi d\xi \qquad (2)$$

where E=relative bed-layer thickness to water depth. Eq. (1) originates from Rouse's sediment concentration distribution; and z=Rouse number that expresses the ratio of the sediment properties to the hydraulic characteristics of the flow (Julien 1995, p. 185). Eq. (2) comes from the product of the logarithmic velocity profile and Rouse sediment concentration distribution. For the purpose of manipulation, the above two integrals can be rearranged as

$$J_1(z) = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^z d\xi - \int_0^E \left(\frac{1-\xi}{\xi}\right)^z d\xi \qquad (3)$$

and

$$J_2(z) = \int_0^1 \left(\frac{1-\xi}{\xi}\right)^z \ln \xi d\xi - \int_0^E \left(\frac{1-\xi}{\xi}\right)^z \ln \xi d\xi \qquad (4)$$

Integral J_1

After using Beta function, Guo and Hui (1991) and Guo and Wood (1995) found that for $z<1$,

$$\int_0^1 \left(\frac{1-\xi}{\xi}\right)^z d\xi = B(1+z,1-z) = \frac{\Gamma(1+z)\Gamma(1-z)}{\Gamma(2)} = \frac{z\pi}{\sin z\pi} \qquad (5)$$

On the other hand, the second term on the right-hand side of Eq. (3) is defined as

$$F_1(z) = \int_0^E \left(\frac{1-\xi}{\xi}\right)^z d\xi \qquad (6)$$

It can be solved using integration by parts as

$$F_1(z) = E\left(\frac{1-E}{E}\right)^z + zF_1(z) + zF_1(z-1) \qquad (7a)$$

or

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$$F_1(z) = -\frac{1}{z-1}\frac{(1-E)^z}{E^{z-1}} - \frac{z}{z-1}F_1(z-1) \tag{7b}$$

Multiple applications of the above recurrence formula results in

$$F_1(z) = -\frac{(1-E)^z}{E^{z-1}}\frac{1}{z-1} + \frac{(1-E)^{z-1}}{E^{z-2}}\frac{z}{z-1}\frac{1}{z-2} + \frac{z}{z-2}F_1(z-2)$$
$$= \frac{(1-E)^z}{E^{z-1}} - z\sum_{k=1}^{\infty}\frac{(-1)^k}{k-z}\left(\frac{E}{1-E}\right)^{k-z} \tag{8}$$

Thus, from Eqs. (3), (5), (6), and (8), one can get J_1 for $z<1$

$$J_1(z) = \frac{z\pi}{\sin z\pi} - \underbrace{\left\{\frac{(1-E)^z}{E^{z-1}} - z\sum_{k=1}^{\infty}\frac{(-1)^k}{k-z}\left(\frac{E}{1-E}\right)^{k-z}\right\}}_{F_1(z)} \tag{9}$$

Similar to Eq. (7*b*), applying integration by parts to Eq. (1), one gets

$$J_1(z) = \frac{1}{z-1}\frac{(1-E)^z}{E^{z-1}} - \frac{z}{z-1}J_1(z-1) \tag{10}$$

Therefore, for $1<z<2$, one obtains

$$J_1(z) = \frac{1}{z-1}\frac{(1-E)^z}{E^{z-1}} - \frac{z}{z-1}\left\{\frac{(z-1)\pi}{\sin(z-1)\pi} - \frac{(1-E)^{z-1}}{E^{z-2}} + (z-1)\sum_{k=1}^{\infty}\frac{(-1)^k}{k-(z-1)}\left(\frac{E}{1-E}\right)^{k-(z-1)}\right\}$$
$$= \frac{1}{z-1}\frac{(1-E)^z}{E^{z-1}} + \frac{z\pi}{\sin z\pi} + \frac{z}{z-1}\frac{(1-E)^{z-1}}{E^{z-2}} - z\sum_{k=1}^{\infty}\frac{(-1)^k}{k-z+1}\left(\frac{E}{1-E}\right)^{k-z+1}$$
$$= \frac{z\pi}{\sin z\pi} - \frac{(1-E)^z}{E^{z-1}} + z\sum_{k=1}^{\infty}\frac{(-1)^k}{k-z}\left(\frac{E}{1-E}\right)^{k-z} \tag{11}$$

which is identical to Eq. (9). Furthermore, one can recognize the self similarity of Eq. (9) for any noninteger value of z.

For any integer $z=n$, a closed solution can be obtained by applying the binomial theorem to the integrand

$$J_1(n) = \int_E^1\left(\frac{1-\xi}{\xi}\right)^n d\xi = \sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!k!}\int_E^1 \xi^{k-n}d\xi$$
$$= \sum_{k=0}^{n-2\geqslant 0}\frac{(-1)^k n!}{(n-k)!k!}\frac{1-E^{k-n+1}}{k-n+1} + n(-1)^{n-1}\int_E^1\xi^{-1}d\xi + (-1)^n\int_E^1 d\xi$$
$$= \sum_{k=0}^{n-2\geqslant 0}\frac{(-1)^k n!}{(n-k)!k!}\frac{E^{k-n+1}-1}{n-k-1} + (-1)^n(n\ln E - E + 1) \tag{12}$$

For example, when $n=3$, it gives

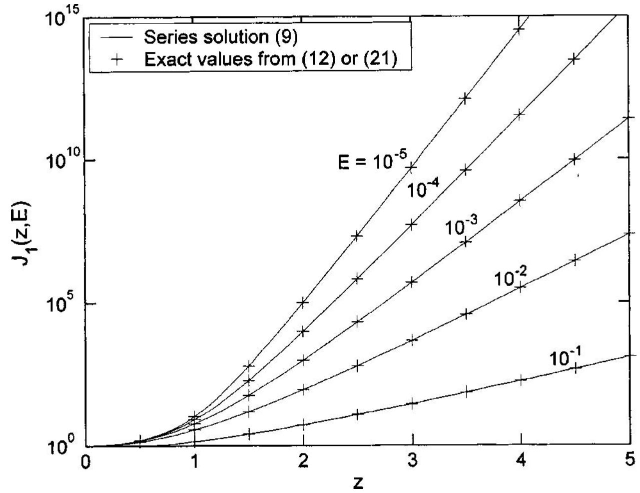

Fig. 1. Plot of integral $J_1\ (z,E)$, Eq. (9)

$$J_1(3) = -3\ln E + \frac{1}{2E^2} - \frac{3}{E} + \frac{3}{2} + E \tag{13}$$

To avoid computational overflow, it is suggested to apply Eq. (9) to any noninteger z value, and use Eq. (12) for any integer z value. In practice, an integer z can be considered $z=n\pm 10^{-3}$. For example, if $z=2.998$, Eq. (9) is used; if $z=2.999$, it can be considered $z\approx 3$ and Eq. (12) is then applied. Besides, from Fig. 1, one can see that Eq. (9) converges to Eq. (12) when z tends to an integer n. In fact, this convergence can also be analytically demonstrated, the proof being beyond the scope of this note.

Integral J_2

Guo and Wood (1995) and Guo (2002) also showed that for $z<1$, one has

$$\int_0^1\left(\frac{1-\xi}{\xi}\right)^z \ln\xi d\xi = \frac{z\pi}{\sin z\pi}[\psi(1-z) - (1-\gamma)]$$
$$= \frac{z\pi}{\sin z\pi}[\psi(z) + \pi\cot z\pi - (1-\gamma)]$$
$$= \frac{z\pi}{\sin z\pi}\left\{\pi\cot z\pi - 1 - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right)\right\} \tag{14}$$

where $\gamma = 0.577\,215\ldots$ = Euler constant; and $\psi(z)$ = psi function, a special function (Andrews 1985). Defining

$$F_2(z) = \int_0^E\left(\frac{1-\xi}{\xi}\right)^z \ln\xi\, d\xi \tag{15}$$

in Eq. (4) and applying integration by parts gives

$$F_2(z) = E\left(\frac{1-E}{E}\right)^z \ln E + zF_2(z-1) + zF_2(z) - F_1(z) \tag{16a}$$

or

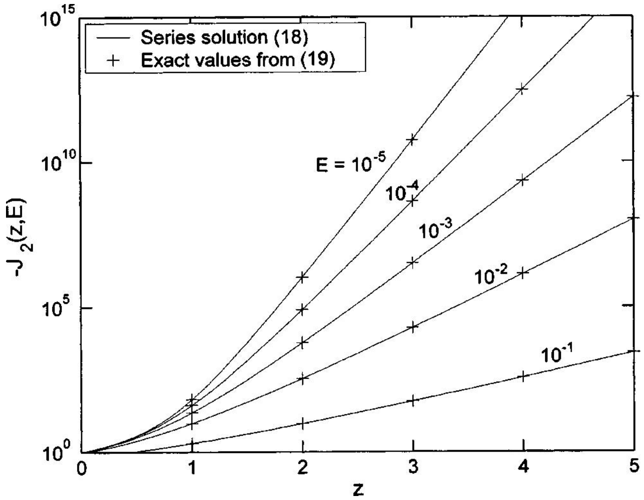

Fig. 2. Plot of integral $J_2\ (z,E)$, Eq. (18)

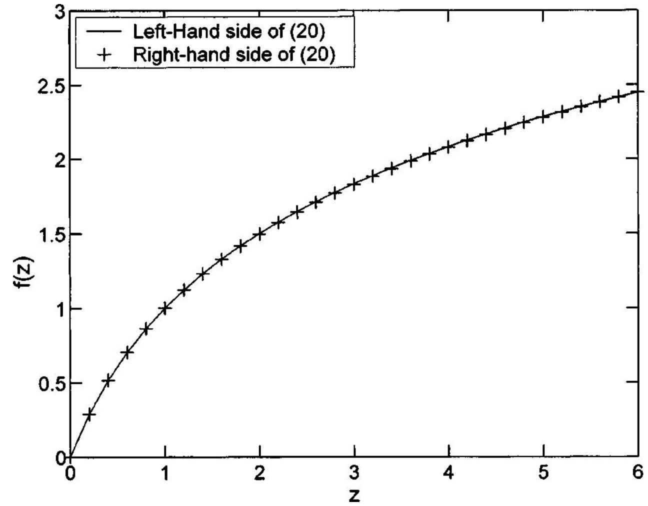

Fig. 3. Approximation of Eq. (20)

$$F_2(z) = -\frac{(1-E)^z}{E^{z-1}}\frac{\ln E}{z-1} - \frac{z}{(z-1)}F_2(z-1) + \frac{F_1(z)}{(z-1)} \quad (16b)$$

This result is similar to Eq. (7*b*). After a complicated derivation, one can show that

$$F_2(z) = F_1(z)\left(\ln E + \frac{1}{z-1}\right) + z\sum_{k=1}^{\infty}\frac{(-1)^k F_1(z-k)}{(z-k)(z-k-1)} \quad (17)$$

in which $F_1(z)$ is estimated by Eq. (8). Finally, Eq. (4) becomes

$$J_2(z) = \frac{z\pi}{\sin z\pi}\left\{\pi\cot z\pi - 1 - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right)\right\} - \underbrace{\left\{F_1(z)\left(\ln E + \frac{1}{z-1}\right) + z\sum_{k=1}^{\infty}\frac{(-1)^k F_1(z-k)}{(z-k)(z-k-1)}\right\}}_{F_2(z)} \quad (18)$$

Like Eq. (9), Eq. (18) is valid for any noninteger z although it is derived for $z<1$. For integer $z=n$, the following closed solution exists

$$\begin{aligned} J_2(n) &= \int_E^1 \left(\frac{1-\xi}{\xi}\right)^n \ln\xi d\xi = \sum_{k=0}^{n}\frac{n!(-1)^k}{(n-k)!k!}\int_E^1 \xi^{k-n}\ln\xi d\xi \\ &= \sum_{k=0}^{n-2\geqslant 0}\frac{(-1)^k n!}{(n-k)!k!}\int_E^1 \xi^{k-n}\ln\xi d\xi - (-1)^n n\int_E^1 \frac{\ln\xi}{\xi}d\xi \\ &\quad + (-1)^n\int_E^1 \ln\xi d\xi \\ &= \sum_{k=0}^{n-2\geqslant 0}\frac{(-1)^k n!}{(n-k)!k!}\cdot\left\{\frac{E^{1+k-n}\ln E}{n-k-1} + \frac{E^{1+k-n}-1}{(n-k-1)^2}\right\} \\ &\quad + (-1)^n\left\{\frac{n}{2}\ln^2 E - E\ln E + E - 1\right\} \end{aligned} \quad (19)$$

For the interest of application, the convergence of Eq. (18) to Eq. (19) is only shown in Fig. 2.

Proposed Algorithm and Convergence

Eqs. (9) and (18) include three infinite series. Series (8) and (17) are rapidly convergent as soon as $k-z>1$, because E^{k-z} quickly tends to zero. In practice, taking the first 10 terms in Eqs. (8) and (17) is accurate enough since there is no sediment transport under $z>10$. The convergence of the first series in Eq. (18) is comparatively slower. For calculation, the following approximation can be used in a program

$$\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{z+k}\right) \equiv f(z) \approx \frac{\pi^2}{6}\frac{z}{(1+z)^{0.7162}} \quad (20)$$

which is shown in Fig. 3 where the maximum relative error is 0.26% for $0\leqslant z\leqslant 6$.

The above analysis can be summarized in the form of a computational algorithm. First, for an integer value z, i.e., $|z-\text{round}(z)|<10^{-3}$, Eqs. (12) and (19) are directly applied. Otherwise, the following algorithm is used.

- *Step 1*: Estimate $F_1(z)$ from Eq. (8) using a maximum of 10 terms, $k=10$.
- *Step 2*: Estimate $J_1(z)$ from Eq. (9).
- *Step 3*: Estimate the first series in Eq. (18) by using the approximation (20).
- *Step 4*: Estimate $F_2(z)$ from Eq. (17) using $k=10$ terms.
- *Step 5*: Estimate $J_2(z)$ from Eq. (18).

A *Fortran* subroutine or *Excel* spreadsheet can be downloaded from http://courses.nus.edu.sg/course/cveguoj/ce5309/pierre.html for the above algorithm. The results of applying this algorithm are plotted in Figs. 1 and 2 where the symbol of a cross indicates the exact values from Eqs. (12) and (19). In addition, the exact values of J_1 for $z=n+1/2$ can be found with *Maple* and are also plotted in Fig. 1. For example,

$$J_1\left(\frac{1}{2}\right) = \frac{\pi}{4} - \frac{1}{2}\sin^{-1}(2E-1) - E\sqrt{\frac{1}{E}-1} \quad (21a)$$

$$J_1\left(\frac{3}{2}\right) = -\frac{3\pi}{4} + \frac{3}{2}\sin^{-1}(2E-1) + (2+E)\sqrt{\frac{1}{E}-1} \quad (21b)$$

$$J_1\left(\frac{5}{2}\right) = \frac{5\pi}{4} - \frac{5}{2}\sin^{-1}(2E-1) + \left(\frac{2}{3E} - \frac{14}{3} - E\right)\sqrt{\frac{1}{E} - 1} \tag{21c}$$

$$J_1\left(\frac{7}{2}\right) = -\frac{7\pi}{4} + \frac{7}{2}\sin^{-1}(2E-1) + \left(\frac{2}{5E^2} - \frac{32}{15E} + \frac{116}{15} + E\right)\sqrt{\frac{1}{E} - 1} \tag{21d}$$

$$J_1\left(\frac{9}{2}\right) = \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}(2E-1) + \left(\frac{2}{7E^3} - \frac{58}{35E^2} + \frac{156}{35E} - \frac{388}{35} - E\right)\sqrt{\frac{1}{E} - 1} \tag{21e}$$

One can see that Eqs. (9) and (18), respectively, converge to Eqs. (12) and (19), the results for integer z values from Eq. (21) also coincide with those from Eq. (9). Thus, one can consider that Eqs. (9) and (18) correctly represent the accurate vales of J_1 and J_2, respectively. The numerical calculation shows that the presented approximations are computationally efficient and can avoid computational overflow. Therefore, they can be incorporated into professional software like *HEC-RAS* or *HEC-6*.

Conclusions

This note presents an effective approximation to Einstein integrals J_1 and J_2 that are valid over the entire range of the Rouse number z and the relative bed-layer thickness E. The approximations can be readily implemented using widespread tools such as programmable calculators, spreadsheets, *Fortran*, or *MatLab*. In particular, it may provide a simple way to incorporate Einstein bed load function into widely used hydraulic software. The numerical experiment shows that the proposed algorithm rapidly converges to the exact values of J_1 and J_2.

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Discussion of "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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The writers are congratulated for developing analytical approximations to Einstein integrals (herein INT_1 and INT_2). It was stated by the writers in their technical note that these algorithms can be incorporated into professional hydraulic software, and the discussers agree on this. However, it is pointed out here that the calculation of INT_1 and INT_2 can be easily incorporated more directly by subroutines or functions for numerical integration (by using any numerical library or by coding one) into any hydraulic and sediment transport model (e.g., Abad 2002). In terms of practicality, there is a need for more simple formulations than those series-based ones presented by the writers. The present discussion focuses on describing practical formulations for calculating the Einstein integrals based on regression analysis. The development of the methodology is derived for use in depth-averaged sediment transport models.

Brief Background

Fig. 1 shows an open-channel configuration. Depending on the vertical location (z), sediment transport can be treated as bed-load ($z<z_b$) or suspended load ($z>z_b$), where z_b is called the reference level or the bed-layer thickness. Suspended sediment load at equilibrium conditions is calculated by using a Rousean profile, as shown below:

$$c(z)=c(z_b)\left[\frac{(H-z)/z}{(H-z_b)/z_b}\right]^{Z_R} \tag{1}$$

where $c(z)$=concentration in the vertical direction; $c(z_b)$=concentration at the reference level (bed-layer thickness); $Z_R=w_s/\kappa u_*$ is the Rouse number; H=water depth; w_s=settling velocity of the particle; κ=Von Karman coefficient ($\sim$0.40 given by experiments); and u_*=shear velocity. A depth-averaged suspended concentration can be calculated by integrating Eq. (1) along the vertical direction as shown in Eq. (2):

$$\bar{C}=\frac{1}{H}\int_{z_b}^{H}c(z)dz=\frac{1}{H}\int_{z_b}^{H}\left[\frac{(H-z)/z}{(H-z_b)/z_b}\right]^{Z_R}dz \tag{2}$$

Using $\delta=z/H$ and $\delta_b=z_b/H$, Eq. (2) can be expressed as

$$\bar{C}=c(z_b)INT_1=c(z_b)\int_{\delta_b}^{1}\left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b}\right]^{Z_R}d\delta \tag{3}$$

Einstein (1950) proposed a relation for the depth-averaged sediment concentration as $\bar{C}=c(z_b)\delta_b I_1/0.216$, where I_1 is given by the well-known Einstein's monographs (Einstein 1950; García 1999, 2005.) Assuming logarithmic velocity profile and identical horizontal velocities for water and sediment, the suspended sediment load can be calculated by using

$$q_s=\int_{z_b}^{H}u(z)c(z)dz=\frac{1}{\kappa}c(z_b)u_*H\left[INT_1\ln\left(30\frac{H}{k_c}\right)+INT_2\right] \tag{4}$$

where k_c represents the composite roughness (i.e., grain resistance and form drag). INT_2 is given by

$$INT_2=\int_{\delta_b}^{1}\left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b}\right]^{Z_R}\ln(\delta)d\delta \tag{5}$$

Again, Einstein (1950) proposed another graph for $-I_2$ in order to calculate INT_2 ($INT_2=\delta_b I_2/0.216$). Guo and Wood (1995) have proposed analytical series-based approximations for INT_1 and INT_2 (called J_1 and J_2 by Guo and Wood, 1995), which are valid for fine sediments ($Z_R<1$). The writers in their recent paper have extended these analytical series-based approximations to be valid for the entire range of Z_R and δ_b. However, their use for practical

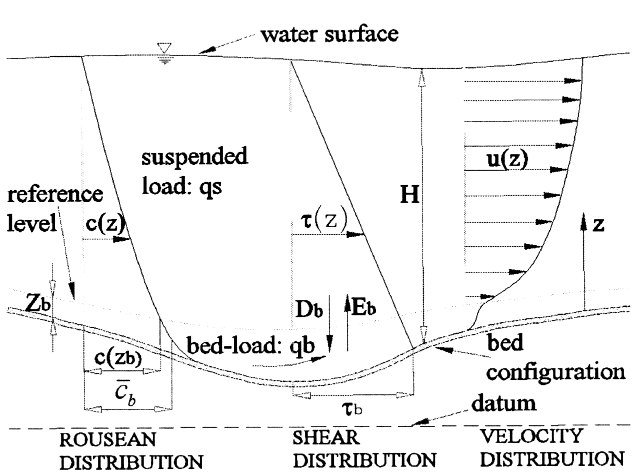

Fig. 1. Open-channel distribution

Table 1. Coefficients for INT_1 Formulation

δ_b	c_0^I	c_1^I	c_2^I	c_3^I	c_4^I	c_5^I	c_6^I	R^2
0.01	1.4852	0.2025	14.087	20.918	−10.91	2.034	−0.1345	1.00
0.02	1.2134	1.9542	10.613	6.0002	−3.6259	0.6938	−0.0462	1.00
0.03	1.1409	2.4266	8.2541	2.4058	−1.7617	0.3474	−0.0234	1.00
0.04	1.1138	2.5982	6.7187	1.029	−1.001	0.2045	0.0139	1.00
0.05	**1.1038**	**2.6626**	**5.6497**	**0.3822**	**−0.6174**	**0.1315**	**−0.0091**	**1.00**
0.06	1.102	2.6809	4.864	0.0422	−0.3989	0.0894	−0.0063	1.00
0.07	1.1048	2.6775	4.2624	−0.1487	−0.2639	0.0629	−0.0045	1.00
0.08	1.1104	2.6636	3.787	−0.2598	−0.1757	0.0454	−0.0033	1.00
0.09	1.1178	2.6448	3.4019	−0.3254	−0.1156	0.0333	−0.0025	1.00
0.1	1.1266	2.6239	3.0838	−0.3636	−0.0734	0.0246	−0.0019	1.00

Table 2. Coefficients for INT_2 Formulation

δ_b	c_0^{II}	c_1^{II}	c_2^{II}	c_3^{II}	c_4^{II}	c_5^{II}	c_6^{II}	R^2
0.01	1.151	2.1787	7.6572	−0.2777	−0.57	0.1424	−0.0105	1.00
0.02	1.1428	2.4442	4.2581	−0.4713	−0.1505	0.0467	−0.0036	1.00
0.03	1.1744	2.4172	3.0015	−0.4405	−0.049	0.0218	−0.0018	1.00
0.04	1.2143	2.364	2.3373	−0.3955	−0.0104	0.0116	−0.001	1.00
0.05	**1.2574**	**2.3159**	**1.9239**	**−0.3558**	**0.0075**	**0.0064**	**−0.0006**	**1.00**
0.06	1.3023	2.2773	1.6411	−0.3228	0.0167	0.0035	−0.0004	1.00
0.07	1.3486	2.2481	1.4351	−0.2955	0.0216	0.0017	−0.0003	1.00
0.08	1.3961	2.2269	1.2782	−0.2728	0.0243	0.0005	−0.0002	1.00
0.09	1.445	2.2125	1.1548	−0.2536	0.0258	−0.0002	−0.0001	1.00
0.1	1.4952	2.2041	1.0552	−0.2372	0.0265	−0.0008	−0.00005	1.00

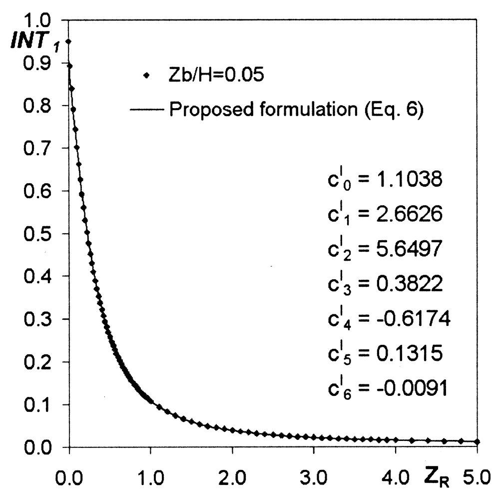

Fig. 2. Comparison between analytical solution and regression analysis for $INT_1(\delta_b=0.05)$

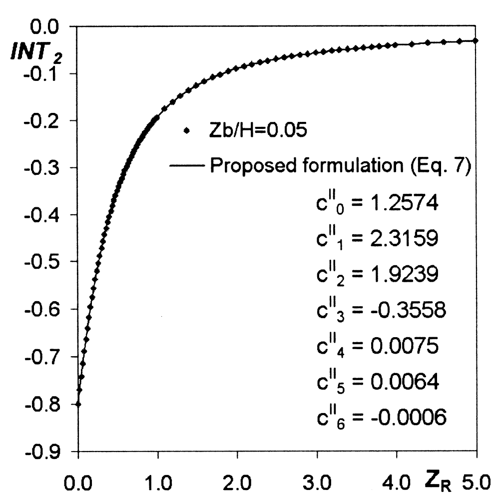

Fig. 3. Comparison between analytical solution and regression analysis for $INT_2(\delta_b=0.05)$

calculations is still limited since the computation of the integrals requires the implementation of a numerical subroutine or function. The effort and time required for incorporating any numerical subroutine or function for the direct integration of INT_1 and INT_2 suggest the necessity of developing more practical approaches that can be implemented more readily into a numerical model. The present discussion attempts to fulfill this last item.

Analysis and Formulations for Integrals INT_1 and INT_2

INT_1 and INT_2 are computed numerically (also computed using similar analytical approximations as proposed by the writers) for different Z_R and δ_b values. Then a sixth-order regression analysis of the results is performed in order to obtain simple formulations for INT_1 and INT_2, as shown by Eqs. (6) and (7) (see coefficients in Table 1 and Table 2):

$$INT_1 = \frac{1}{c_0^I + c_1^I Z_R + c_2^I Z_R^2 + c_3^I Z_R^3 + c_4^I Z_R^4 + c_5^I Z_R^5 + c_6^I Z_R^6} \quad (6)$$

$$INT_2 = \frac{-1}{c_0^{II} + c_1^{II} Z_R + c_2^{II} Z_R^2 + c_3^{II} Z_R^3 + c_4^{II} Z_R^4 + c_5^{II} Z_R^5 + c_6^{II} Z_R^6} \quad (7)$$

The writers have presented series-based approximations of the Einstein integrals. The discussers have developed additional formulations of these integrals with their practical application in mind. The proposed alternative approximations need some coefficients for different values of δ_b; however, in most existing formulas for estimating sediment entrainment or near-bed concentrations under equilibrium conditions, the reference level $\delta_b = z_b/H$ is taken to be 0.05 (Itakura and Kishi 1980; Celik and Rodi 1984; Akiyama and Fukushima 1986; García and Parker 1991, among others), which involves seven coefficients for each integral. Figs. 2 and 3 show comparisons of the exact solutions (dots) against the proposed practical formulation (continuous line) for INT_1 and INT_2, respectively. Good agreement was found by using these practical formulations.

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Discussion of "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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The discussers have found an explicit approximation for the Einstein Integrals which result in a discrepancy less than 1% to the approximation of the writers, though the approximation has greater errors when the dimensionless reference height becomes bigger than *0.01 for the first integral* and *0.001 for the second integral*. However, a huge fraction of natural conditions is covered by this approximation. The calculation speed is more than two orders faster than the approximation of the writers and nearly twice as fast as the recurrence formula suggested by Guo et al. (1996), which is the foundation for the suggested formula. The combination of the explicit approximation and the recurrence formula is suggested as an optimal algorithm for the estimation of both integrals. In the following, the recurrence formula of Guo et al. (Approximation II) and the explicit approximation (Approximation III) by the discusser will be compared with respect to performance and accuracy with the actual formulation of the writers (Approximation I). The derivation of the explicit approximation for the integral $J1$ starts with the identical idea as the writers', which is multiple application of the recurrence formula (1)

$$J1_{(\xi_b,z)} = \left(\frac{1}{z-1}\right)\cdot\left[\frac{(1-\xi_b)^z}{\xi_b^{z-1}}\right] - \left(\frac{1}{z-1}\right)\cdot J1_{(\xi_b,z-1)} \tag{1}$$

In Eq. (1) $\xi_b = a/h$, with a the reference height, h the water depth, and z the Rouse number. To get an explicit formula for the calculation of $J1$, Eq. (1) was expanded three times in z

$$J1_{(\xi_b,z)} = \frac{z\cdot\pi}{\sin[z\cdot\pi]} - \frac{\xi_b^{1-z}}{1-z} \tag{2}$$

The explicit approximation was obtained with the aid of Eq. (2), which is an approximation for $J1$ when $z<1$ (see Guo et al.). This equation leads to a discrepancy in the solution of the writers, which is less than 1% in if ξ_b is smaller than 0.01 (Fig. 1).

$$\begin{aligned} J1_{(\xi_b,z)} = {} & \left(\frac{1}{z-1}\right)\cdot\left[\frac{(1-\xi_b)^z}{\xi_b^{z-1}}\right] \\ & - \left(\frac{z}{z-1}\right)\cdot\Bigg\{\left(\frac{1}{z-2}\right)\cdot\left[\frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}}\right] \\ & - \left(\frac{z-1}{z-2}\right)\cdot\Bigg\{\left(\frac{1}{z-3}\right)\cdot\left[\frac{(1-\xi_b)^{z-2}}{\xi_b^{z-3}}\right] \\ & - \left(\frac{z-2}{z-3}\right)\cdot\left[\frac{(z-3)\cdot\pi}{\sin[(z-3)\cdot\pi]} - \frac{\xi_b^{4-z}}{4-z}\right]\Bigg\}\Bigg\} \end{aligned} \tag{3}$$

The explicit approximation of the second Einstein Integral is also based on the publication of Guo et al., but the accuracy is not as good as the approximation of the first Einstein Integral. The starting point for the derivation of the explicit approximation for $J2$ is again the recurrence formula given by Guo et al. for $J2$:

$$J2_{(\xi_b,z)} = \frac{1}{z-1}\cdot\left[\left(\frac{(1-\xi_b)^z}{\xi_b}\right)\cdot\ln(\xi_b) - z\cdot J2_{(\xi_b,z-1)} + J1_{(\xi_b,z)}\right] \tag{4}$$

The explicit approximation is achieved in a similar way as for $J1$. Guo et al. have suggested for $z<1$ the following approximation for $J2$:

$$J2_{(\xi_b,z)} = -\frac{z\cdot\pi\cdot f_{(z)}}{\sin(z\cdot\pi)} - \frac{\xi_b^{z-1}}{1-z}\cdot\log(\xi_b) + \frac{\xi_b^{z-1}}{(1-z)^2}$$

$$f_{(z)} = (1-\gamma) - \log(2-z) + \frac{1}{1-z} + \frac{1}{2\cdot(1-z)} + \frac{1}{24\cdot(2-z)^2} \tag{5}$$

Here $f(z)$=approximation of a more complicated functional in terms of Gamma functions given by Guo and Wood (1995). An explicit approximation for $J2$ was found after double expansion of Eq. (4) and replacement of the terms $J1$ and $J2$ in Eq. (6), with the functions summarized in Eq. (7). The final equation shows a discrepancy to the solution of the writers which is less than 1% for a reference height that is smaller than 0.001 (Fig. 2)

$$\begin{aligned} J2_{(\xi_b,z)} = {} & \left(\frac{1}{z-1}\right)\cdot\Bigg\{\log(\xi_b)\cdot\frac{(1-\xi_b)^z}{\xi_b^{z-1}} \\ & - z\cdot\Bigg[\left(\frac{1}{z-2}\right)\cdot\Bigg(\log(\xi_b)\cdot\frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}} \\ & - (z-1)\cdot J2_{(\xi_b,z-3)}\cdot J1_{(\xi_b,z-2)}\Bigg)\Bigg] + J1_{(\xi_b,z)}\Bigg\} \end{aligned} \tag{6}$$

The approximation of the function $f_{(z)}$ is not straightforward as the second term becomes undefined when the argument is negative ($z>4$). Numerical experiments with (6) have shown that a good approximation can be achieved if the absolute argument in the log function is calculated.

$$J2_{(\xi_b,z-3)} = -\frac{(z-2)\cdot\pi\cdot f_{(z)}}{\sin((z-2)\cdot\pi)} - \frac{\xi_b^{3-z}}{3-z}\cdot\log(\xi_b) + \frac{\xi_b^{3-z}}{(3-z)^2}$$

$$\begin{aligned} J1_{(\xi_b,z-2)} = {} & \left(\frac{1}{z-2}\right)\cdot\left[\frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}}\right] \\ & - \left(\frac{z-1}{z-2}\right)\cdot\left[\frac{(z-2)\cdot\pi}{\sin[(z-2)\cdot\pi]} - \frac{\xi_b^{3-z}}{3-z}\right] \end{aligned}$$

$$f_{(z)} = (1-\gamma) - \log(|4-z|) + \frac{1}{3-z} + \frac{1}{2\cdot(4-z)} + \frac{1}{24\cdot(4-z)^2} \tag{7}$$

The results of the integrals $J1$ and $J2$ obtained with the approximation of the writers have been compared with the recurrence formula of Guo et al. and the explicit approximation suggested by the discussers

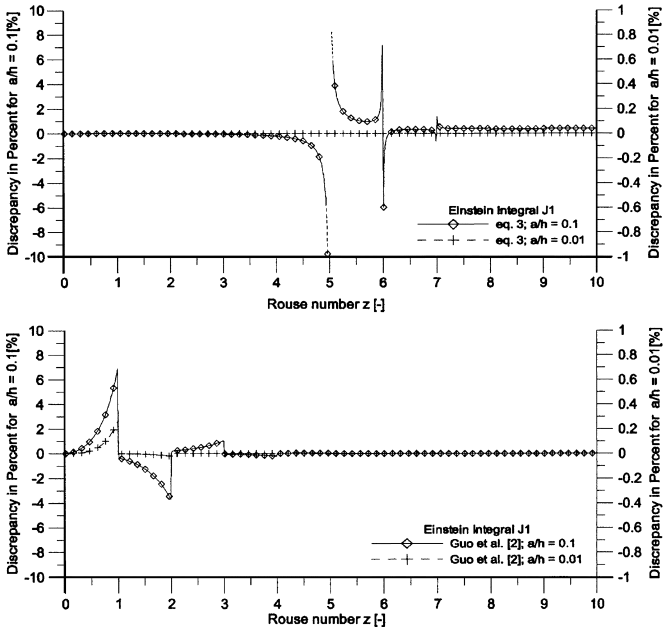

Fig. 1. Discrepancy in percent between the suggested explicit approximation (top) and the recurrence formula [2] (bottom) in comparison to the approximation of the writers for $J1$. Discontinues lines indicates that the curve is limited to the range of the axis.

$$
z = \{z \in (0.01, 0.02, \ldots 10.0)\}
$$

$$
\xi_b = \{\xi_b \in (10^{-5.0}, 10^{-4.9}, \ldots 10^{-1.0})\} \tag{8}
$$

Both approaches lead to rather big errors, if the Rouse number z approaches integer values and if the dimensionless reference height ξ_b approaches unity. When the reference height a becomes small in comparison to h, the discrepancies disappear. The reason is that the recurrence formulas have been derived by Guo et al. under the assumption that a is small in comparison to h.

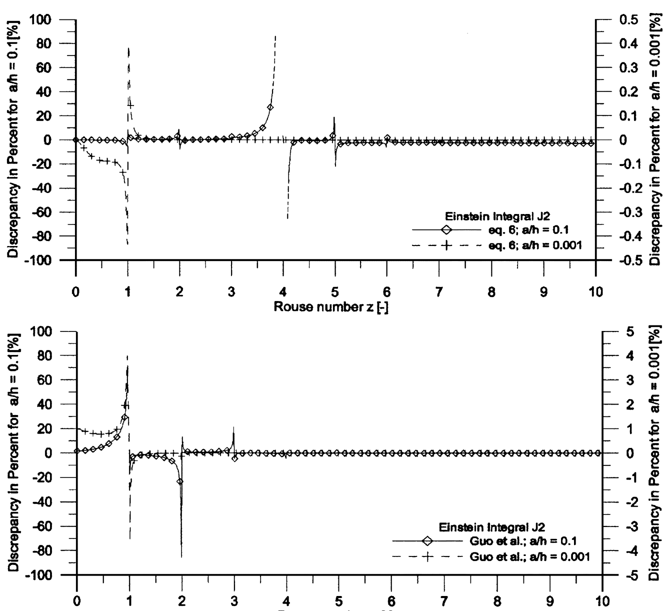

Fig. 2. Discrepancy in percent between the suggested explicit approximation (top) and the recurrence formula [2] (bottom) in comparison to the approximation of the writers for $J1$

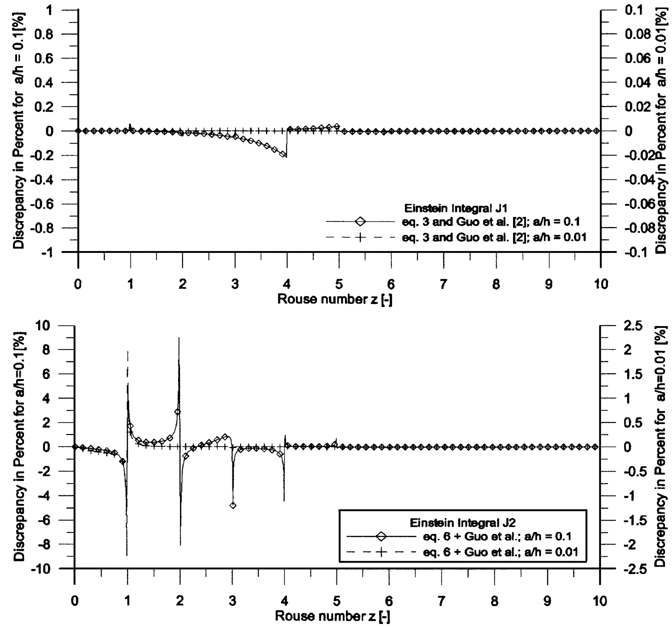

Fig. 3. Error in percent between the suggested optimal approximation algorithm and the results of the writers

To achieve optimal performance and accuracy, the recurrence formula and the explicit approximation can be applied for the distinct areas in the co-domain in which these algorithms result in small errors in comparison to the exact solution. In Fig. 3, the results for such an algorithm are presented. It can be seen that for the integral $J1$ the discrepancy reduces to less than 1% for $J1$. For $J2$ the discrepancies are still one order higher than for the integral $J1$, though significantly reduced in comparison to Fig. 2. The error for $J2$ growths are maximal for $z=1$ and $\xi_b > 0.01$, but it is still less than 2.5% while $\xi_b < 0.01$. The most accurate approximation of the writers can be used if higher accuracy is anticipated in the designated regions.

We have summarized the computation time for the calculation of 10 times the co-domain [Eq. (8) for these three different approximations (see Fig. 4). The approximation of the writers is the slowest one due to its numerical complexity, though it is the most exact one. The explicit approximation is nearly twice as fast as the recurrence formula and more than two orders faster than the

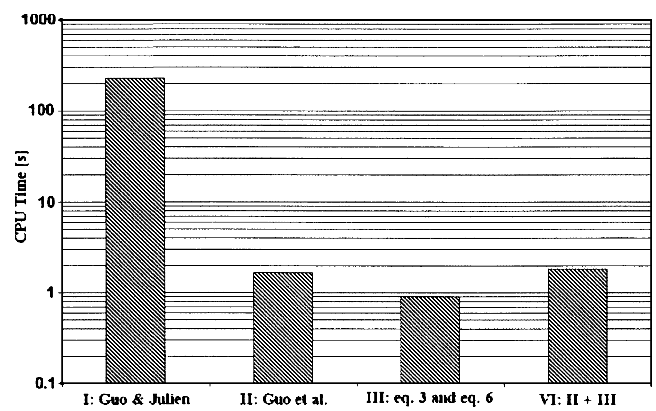

Fig. 4. Calculation time of the various approximations for the calculation of 10 times the co-domain

approximation of the writers. The combination of Approximations II and III is still more than two orders faster than the recent one suggestd by the writers.

Conclusions

An optimal scheme for calculating the Einstein Integrals has been proposed on the foundation of the work of Guo et al. and the suggested explicit approximation. The new suggested algorithm can be efficiently used in morphodynamic models like TIMOR (TIdal MORphodynamics; Zanke 2002), which are based on the solution of vertical integrated flow models. In such kinds of applications the suspended sediment transport rates have to be calculated for every time-step, grid point, and grain size, which leads to high calculation times. Therefore the improvement of the calculation speed and accuracy has direct impact on the performance of the morphodynamic model. This study was only possible due to the work of Prof. Junke Guo in the past 10 years concerning the derivation of an approximation for the Einstein Integrals. In the latest publication, discussed here, the writers found the most exact approximation of these equations to date, which is considered to be an important step in science.

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Discussion of "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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The writers have proposed infinite series expressions for evaluating the Einstein integrals. The discusser would like to mention the following points regarding the note:

1. Eq. (3) is only valid for $z<1$ because for $z\geqslant 1$ both the integrals on the right-hand side are *divergent*. Similarly, Eqs. (7) and (8) are valid only for $z<1$. While this limitation was stated explicitly before Eqs. (5) and (9), it should have been mentioned at other relevant places also.
2. While the range of z has been stated ($z<10$) in the note, that for E has not been mentioned. The value of E used by the writers ranges from 0.00001 to 0.1. Obviously, if E is higher than 0.1, the convergence of the series in Eqs. (9) and (18) would be quite slow. In fact, if $E>0.5$, Eq. (8) will not be valid since $F_1(-\infty)\neq 0$ for $E/(1-E)>1$. Such high values of E may be practically impossible, but mathematical rigor demands a passing mention of the fact. The sentence in the Conclusions section stating, "…*valid over the entire range of the Rouse number z and the relative bed-layer thickness E*," should therefore be qualified with additional information.
3. While integer (and $n+1/2$) values of z may be useful for comparing the proposed algorithm with exact solutions, it is very unlikely that the actual value of z would be an *exact* integer. The discusser feels that the paragraph following Eq. (13) is therefore not really appropriate. In fact, it may be more desirable to have a single expression [Eq. (9)] for all values of z with the provision that if z is an exact integer, its value would be taken as $z\pm 0.001$!
4. In the paragraph above Eq. (20), rapid convergence of series (8) and (17) should be based on rapid approach to zero of $[E/(1-E)]^{k-z}$, and not E^{k-z}. As discussed in point 2 above, if $E>0.5$, E^{k-z} will still approach zero rapidly but the series will not converge.
5. While the writers mention the limit $z=10$ before Eq. (20), the maximum relative error for the approximation is stated only for $z<6$. It was found that the error increases with further increase in z and becomes more than 0.8% for $z=10$. While this may be acceptable for practical purposes, the approximation may be considerably improved by considering the limiting behavior of the series in Eq. (20). The following approximation was derived with the use of the limiting behavior and was found to have a maximum error of only 0.03% over the entire range $z>0$:

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{z+k}\right)=\ln(1+1.781z)-\frac{0.1361z}{(1+1.284z)^{2.150}} \quad (1)$$

6. An alternative methodology for deriving series expressions for J_1 is described below: Since $(1-\xi)/\xi$ is less than 1 for $\xi>0.5$ and more than 1 for $\xi<0.5$, we write

$$J_1(z,E)=\int_E^{0.5}\left(\frac{1-\xi}{\xi}\right)^z d\xi+\int_{0.5}^{1}\left(\frac{1-\xi}{\xi}\right)^z d\xi \quad (2)$$

Making the substitution $x=\xi/(1-\xi)$ in the first integral and $x=(1-\xi)/\xi$ in the second integral, we get

$$J_1(z,E)=\int_{E_*}^{1}x^{-z}(1+x)^{-2}dx+\int_0^1 x^z(1+x)^{-2}dx$$
$$=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}i[1-E_*^{i-z}]}{i-z}+\sum_{i=1}^{\infty}\frac{(-1)^{i-1}i}{i+z} \quad (3)$$

where $E_*=E/(1-E)$. Both these series, however, converge slowly. Also, for integer z values, the computations have to make use of the fact that $\lim_{p\to 0}(x^p-1)/p=\ln x$. To avoid the slow convergence, the following approximation was obtained with a maximum error of 0.18% using the data for $0\leqslant z\leqslant 5$ (in steps of 0.5) and $-5\leqslant \log E\leqslant -1$ (in steps of 1):

$$J_1(z,E)=-\frac{E_*^{1-z}-1}{1-z}+2.061\frac{E_*^{2-z}-1}{2-z}-1.385\frac{E_*^{2.6-z}-1}{2.6-z}+\frac{0.3327}{0.6703+z} \quad (4)$$

Similarly, $J_2(z,E)$ was approximated with a maximum error of 0.03% as

$$J_2(z,E)=\frac{E_*^{1-z}[1-(1-z)\ln E_*]-1}{(1-z)^2}-1.903\frac{E_*^{2-z}[1-(2-z)\ln E_*]-1}{(2-z)^2}+2.022\frac{E_*^{2.6-z}[1-(2.6-z)\ln E_*]-1}{(2.6-z)^2}-\frac{0.2914}{1.652+z} \tag{5}$$

Again, in the unlikely event of z being exactly equal to 1, 2, or 2.6, we make use of the fact that

$$\lim_{p\to 0}\frac{x^p[1-p\ln x]-1}{p^2}=-\frac{(\ln x)^2}{2}$$

The discusser feels that Eqs. (4) and (5) above would perform better than the writers' Eqs. (9) and (18) for computation of the Einstein integrals.

Closure to "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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We would like to thank all the discussers for their constructive comments and for providing alternative approximations of the Einstein integrals. It is commonly agreed that these two integrals remain elusive to exact solutions. However, a lot of progress has been made in developing fast and accurate approximations. Abad and Garcia emphasize the need for simpler algorithms and propose a polynomial approximation. Srivastava provides keen insight into the convergence of the series and offers improvements. Roland et al. present an error analysis as well as a valuable comparison of the computational time of various algorithms. Besides polynomial approximations and different series expansions, other e-mail communications by A. R. Kacimov pointed to the possible use of hypergeometric series as well as software packages like *MatLab*, *Maple*, and *Mathematica*. Four main issues are raised in the discussions and they are addressed in the following sequence: (1) convergence; (2) algorithm efficiency; (3) accuracy; and (4) computational time. This closure also includes a comparative analysis of the different algorithms and an application example on the Missouri River.

First, regarding convergence, Srivastava correctly points out that the proposed series are not convergent when the bed layer thickness E is greater than 0.5. Well, this is a trivial case because when $E>0.5$, sediment is exclusively transported as bedload and the integration is not required. For most practical applications, the value of E is relatively small. For a typical laboratory flume experiment, given a grain diameter of $D=1$ mm and a flow depth of $h=10$ cm, the value of E is 0.02. The value of E for field applications is even smaller, as per the example below.

Second, all the discussions express concerns about the computational efficiency of the proposed method. It was readily acknowledged in the Technical Note that the first series of J_2 in Eq. (18) converges slowly. Calculation times of the order of a second were not considered excessive, and accuracy was preferred over CPU time. In view of the discussions, however, it is now possible to substantially improve computational efficiency. The results of an earlier formulation proposed by Guo and Wood can be expanded and simplified to replace the former Eq. (14) with

$$\begin{aligned}\int_0^1\left(\frac{1-\xi}{\xi}\right)^z\ln\xi d\xi &=-\frac{z\pi}{\sin z\pi}[(1-\gamma)-\psi(1-z)]\\ &=-\frac{z\pi}{\sin z\pi}\left[(1-\gamma)-\psi(n+1-z)+\sum_{k=1}^{n}\frac{1}{k-z}\right]\\ &=-\frac{z\pi}{\sin z\pi}\left[(1-\gamma)-\ln(n-z)-\frac{1}{2(n-z)}+\frac{1}{12(n-z)^2}\right.\\ &\qquad\left.-\frac{1}{120(n-z)^4}+\cdots+\sum_{k=1}^{n}\frac{1}{k-z}\right]\end{aligned} \tag{1}$$

in which $\gamma=0.5772156649015328606065\ldots$ and $\psi(x)$ is the psi function, a special function. For calculations, it is suggested that n take as $n=\text{ceil}(z)+2$, in which $\text{ceil}(z)$ means the ceiling of the value of z. For example, if $z=0.1$, $n=\text{ceil}(0.1)+2=3$; if $z=3.7$, $n=\text{ceil}(3.7)+2=6$. Although it is derived for $z<1$, Eq. (1) above is valid for any noninteger value of z. One can demonstrate that Eq. (1) requires fewer terms than Eq. (14) and converges rapidly.

After replacing Eq. (14) with Eq. (1), the proposed algorithm can be slightly modified. For noninteger z, i.e., $|z-\text{round}(z)|>0.005$,

- *Step 1*: Estimate $F_1(z)$ from Eq. (8) using a maximum $k=\text{ceil}(z)+4$.
- *Step 2*: Estimate $J_1(z)$ from Eq. (9).
- *Step 3*: Estimate Eq. (1) above using a value $n=\text{ceil}(z)+2$.
- *Step 4*: Estimate $F_2(z)$ from Eq. (17) using a maximum $k=\text{ceil}(z)+2$.
- *Step 5*: The value of $J_2(z)$ is then obtained by subtracting the result of *Step 4* from the result of *Step 3*.

For integer z values, one can use the same steps as those in the technical note or a recurrent formula like Eq. (10). A program in both *FORTRAN* (einstein.f90) and *MatLab* (einstein.m) for the above algorithm, together with alternatives proposed by the discussers, can be downloaded from ⟨http://myweb.unomaha.edu/~junkeguo⟩ or ⟨http://www.engr.colostate.edu/%7Epierre/ce_old/Projects/index.html⟩. The software programs of readers interested in sharing their source codes will be made available at the same site.

Third, in terms of accuracy, the formulation of Roland et al. is a combination of the three versions of Guo and Wood, Guo et al., and Guo and Julien. There is, however, some concern regarding their statement "the error for J_2 becomes maximal for $z=1$ and E or $\xi_b>0.01$ but it is still less than 10%." Values of $z=1$ are not unusual in practice and 10% looks like a disquietly large error. Roland kindly provided a *FORTRAN* code of his algorithm, but we experienced difficulties running his software and replicating his results. A comparison of several algorithms is presented in this section. Values of $E=0.1$ and $z=0.55$, 1.55, 2.55, 3.55, and 4.55 are considered for the calculations. This large value for E is se-

Table 1. Comparison of Various Algorithms at $E=0.1$

Method	$z=0.55$	$z=1.55$	$z=2.55$	$z=3.55$	$z=4.55$
		(a) Value of J_1			
Exact value	0.97458	2.7326	13.002	77.623	519.35
Guo and Julien (Technical Note)	0.97458	2.7326	13.002	77.623	519.35
Guo and Julien Eq. (1)	0.97458	2.7326	13.002	77.623	519.35
Abad and Garcia	0.97433	2.7350	12.997	77.620	519.19
Srivastava	0.97362	2.7313	12.982	77.604	519.81
		(b) Value of J_2			
Exact value	−1.1201	−4.4912	−24.513	−155.85	−1078.9
Guo and Julien (Technical Note)	−1.1201	−4.4913	−24.513	−155.85	−1078.9
Guo and Julien Eq. (1)	−1.1201	−4.4913	−24.513	−155.85	−1078.9
Abad and Garcia	−1.1200	−4.4921	−24.513	−155.84	−1078.0
Srivastava	−1.1258	−4.5535	**−28.742**	−155.93	−1107.8

Note: The exact values are numerical integrations with Maple.

lected here because Roland et al. showed that it corresponds to larger differences between approximations and exact solutions. The results obtained for the two integrals J_1 and J_2 using four different algorithms (from the Technical Note; Eq. (1); Abad and Garcia; and Srivastava) are summarized in Table 1. They are compared with the exact values obtained from numerical integration in Maple. It is concluded that like the results from the original note, the improvement with Eq. (1) is accurate. Other methods also produce fine results. However, the method of Srivastava can generate errors larger than the value of 0.03% cited in his discussion. For instance, Table 1 shows a 17% error for J_2 at $z=2.55$. Things get worse around $z=2.6$ with errors as large as 800% when $z=2.6-0.001=2.599$. A positive value for $J_2=186.19$ is also incorrectly obtained when $z=2.6+0.001=2.601$.

Fourth, regarding computational time, the CPU times of three algorithms are considered. The calculations were performed with *MatLab* and *FORTRAN* on a Dell Notebook. The results of the methods are very comparable at about 0.015625 s. In addition, one can produce Figs. 1 and 2 in the Technical Note in only 0.875 s. It is fair to say that these algorithms to solve the two Einstein integrals are fast and accurate. Further comparison of the algorithms is presented in the following field application to the Missouri River.

Example: A calculation example with field data is presented here. The site of the Missouri River near Omaha, Neb., is selected for sediment transport calculations using Einstein's method. The main parameters are slope $S=0.00012$, flow depth $h=7.8$ ft$=2.38$ m, and water temperature $T=7°$C. The measured velocity profile u and suspended sand concentration c for the fraction passing a 0.105 mm sieve and retained on a 0.074 mm sieve are shown in Table 2 (from Julien 1995, p. 202). The assignment is to calculate the unit sediment discharge for this size fraction.

Solution: Several algorithms are used for the determination of J_1 and J_2. We use the methods of Srivastava, Abad and Garcia, and Eq. (1) for comparisons with the exact solution. The measured velocity profile and the suspended sediment concentration distribution are fitted as follows:

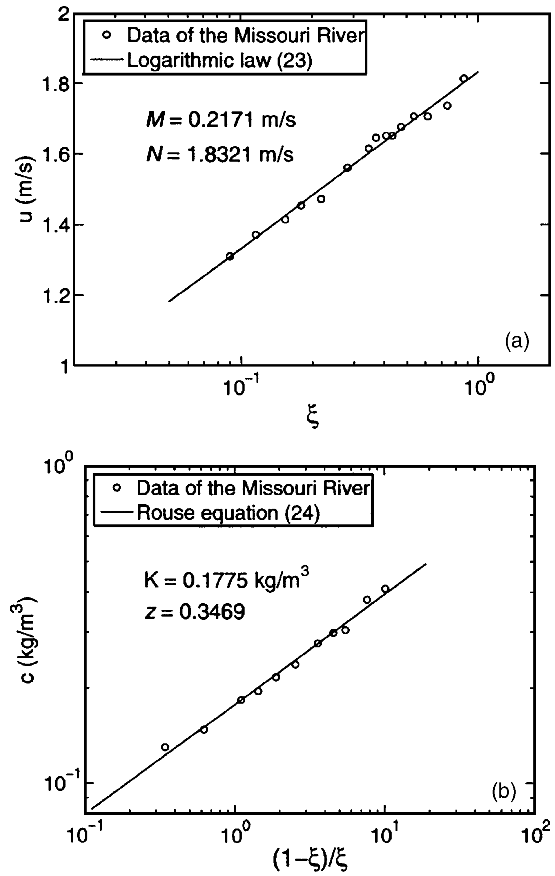

Fig. 1. Curve-fittings of velocity and concentration distributions

Table 2. Measurements of Distributions of Velocity and Concentration in Missouri River

ξ	u (m/s)	c (kg/m^3)
0.090	1.31	0.411
0.115	1.37	0.380
0.154	1.41	0.305
0.179	1.45	0.299
0.218	1.47	0.277
0.282	1.56	0.238
0.346	1.62	0.217
0.372	1.65	—
0.410	1.65	0.196
0.436	1.65	—
0.474	1.68	0.184
0.538	1.71	—
0.615	1.71	0.148
0.744	1.74	0.130
0.872	1.81	—
1.000	—	—

$$u = M \ln \xi + N \quad (2)$$

$$c = K\left(\frac{1-\xi}{\xi}\right)^z \quad (3)$$

in which ξ=distance relative to the flow depth from the bed; and M, N, K, and z=fitting constants from the measurements. Fig. 1 gives $M=0.2171$ m/s, $N=1.8321$ m/s, $K=0.1775$ kg/m^3, and $z=0.3469$. With Eqs. (2) and (3), the Einstein bed-load function can be written as

$$q_T = KMh\left[4.61\frac{(1-E)^z}{E^{z-1}} + \frac{N}{M}J_1(z,E) + J_2(z,E)\right] \quad (4)$$

in which q_T=unit sediment discharge including bed load, the 1st term, and suspended load, the 2nd and 3rd terms. Given the median sediment size $d_m=(0.105+0.074)/2=0.0895$ mm, the bed-layer thickness is then $a=2d_m=0.179$ mm, and the relative thickness $E=a/h=7.5\times10^{-5}$. Compared with the exact value of 0.7520 kg/s.m from Maple, the improved algorithm with Eq. (1) gives $q_T=0.7520$ kg/s.m with a CPU time 0.015 s; and Srivastava's method gives $q_T=0.7496$ kg/s.m, also with a CPU time 0.015 s. The method of Abad and Garcia with the coefficients at the lowest value $E=\delta_b=0.01$ gives $q_T=4.3667$ kg/s.m with the same CPU time. This example shows that Eq. (1) and Srivastava's method are comparable. The method of Abad and Garcia leads to a 580% error in this case because of the small values of E. It is clear that a better approximation of the 14 coefficients would improve the method of Abad and Garcia when $\delta_b<0.01$. The *FORTRAN* program (missouri.f90) and *MatLab* program (missouri.m) for this example are available at the previously mentioned Web site.

In summary, the insightful discussions lead us to a substantially improved solution for J_2 in Eq. (1). This algorithm is quick, accurate, efficient, and convergent. The other methods proposed in the discussions are also quite good although some deficiencies are observed. For instance, the method of Abad and Garcia can lead to substantial errors when $E<0.01$ or $\delta_b<0.01$. The method of Srivastava is also prone to large errors around $z=2.6$. Software and algorithms can be downloaded and shared at the writers' Web sites. In view of such excellent discussions of this article, H. A. Einstein would certainly appreciate the renewed interest in the use of his method.

References

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Discussion of "Simulation of Flow and Mass Dispersion in Meandering Channels" by Jennifer G. Duan

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This paper is of great interest to the community of hydraulic engineering. The writer has developed an enhanced two-dimensional numerical model for simulating flow hydrodynamics and mass transport in meandering channels. The dispersion terms in momentum equations play an important role in the writer's model. The dispersion terms result from the discrepancy between the depth-averaged velocity and the actual velocity. The writer used the logarithmic law to describe the streamwise velocity profile

$$\frac{u_l}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z}{z_0}\right) \quad (1)$$

where u_l=streamwise velocity; u_*=shear velocity; κ=Von Kármán's constant (=0.4); z=vertical coordinate; and z_0=constant (a certain distance from the wall).

Zero-Velocity Level

The logarithmic velocity profile shows that the flow has zero velocity at $z=z_0$ and obviously requires a condition of $z\geqslant z_0$. The constant z_0, also called the zero-velocity level, is of the same order of magnitude as the viscous sublayer thickness and is a function of whether the boundary is hydraulically smooth or rough.

The constant z_0 is an important parameter in the dispersion terms of the momentum equations, as shown in Eqs. (15) to (17) of the original paper. To calculate z_0, the writer used Eq. (2), which contains three relations for flow in three boundary regimes, namely, those of the hydraulically smooth regime ($\mathsf{R}_e\leqslant5$), the completely rough regime ($\mathsf{R}_e\geqslant70$), and the transitional regime ($5<\mathsf{R}_e<70$), according to the magnitude of roughness Reynolds number R_e, defined as $\mathsf{R}_e=u_*k_s/\nu$, where ν=kinetic viscosity and k_s=roughness height