DISCUSSIONS AND CLOSURES

Discussion of "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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The writers are congratulated for developing analytical approximations to Einstein integrals (herein INT_1 and INT_2). It was stated by the writers in their technical note that these algorithms can be incorporated into professional hydraulic software, and the discussers agree on this. However, it is pointed out here that the calculation of INT_1 and INT_2 can be easily incorporated more directly by subroutines or functions for numerical integration (by using any numerical library or by coding one) into any hydraulic and sediment transport model (e.g., Abad 2002). In terms of practicality, there is a need for more simple formulations than those series-based ones presented by the writers. The present discussion focuses on describing practical formulations for calculating the Einstein integrals based on regression analysis. The development of the methodology is derived for use in depth-averaged sediment transport models.

Brief Background

Fig. 1 shows an open-channel configuration. Depending on the vertical location (z), sediment transport can be treated as bed-load ($z<z_b$) or suspended load ($z>z_b$), where z_b is called the reference level or the bed-layer thickness. Suspended sediment load at equilibrium conditions is calculated by using a Rousean profile, as shown below:

$$c(z)=c(z_b)\left[\frac{(H-z)/z}{(H-z_b)/z_b}\right]^{Z_R} \tag{1}$$

where $c(z)$=concentration in the vertical direction; $c(z_b)$=concentration at the reference level (bed-layer thickness); $Z_R=w_s/\kappa u_*$ is the Rouse number; H=water depth; w_s=settling velocity of the particle; κ=Von Karman coefficient ($\sim$0.40 given by experiments); and u_*=shear velocity. A depth-averaged suspended concentration can be calculated by integrating Eq. (1) along the vertical direction as shown in Eq. (2):

$$\bar{C}=\frac{1}{H}\int_{z_b}^{H}c(z)dz=\frac{1}{H}\int_{z_b}^{H}\left[\frac{(H-z)/z}{(H-z_b)/z_b}\right]^{Z_R}dz \tag{2}$$

Using $\delta=z/H$ and $\delta_b=z_b/H$, Eq. (2) can be expressed as

$$\bar{C}=c(z_b)INT_1=c(z_b)\int_{\delta_b}^{1}\left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b}\right]^{Z_R}d\delta \tag{3}$$

Einstein (1950) proposed a relation for the depth-averaged sediment concentration as $\bar{C}=c(z_b)\delta_b I_1/0.216$, where I_1 is given by the well-known Einstein's monographs (Einstein 1950; García 1999, 2005.) Assuming logarithmic velocity profile and identical horizontal velocities for water and sediment, the suspended sediment load can be calculated by using

$$q_s=\int_{z_b}^{H}u(z)c(z)dz=\frac{1}{\kappa}c(z_b)u_*H\left[INT_1\ln\left(30\frac{H}{k_c}\right)+INT_2\right] \tag{4}$$

where k_c represents the composite roughness (i.e., grain resistance and form drag). INT_2 is given by

$$INT_2=\int_{\delta_b}^{1}\left[\frac{(1-\delta)/\delta}{(1-\delta_b)/\delta_b}\right]^{Z_R}\ln(\delta)d\delta \tag{5}$$

Again, Einstein (1950) proposed another graph for $-I_2$ in order to calculate INT_2 ($INT_2=\delta_b I_2/0.216$). Guo and Wood (1995) have proposed analytical series-based approximations for INT_1 and INT_2 (called J_1 and J_2 by Guo and Wood, 1995), which are valid for fine sediments ($Z_R<1$). The writers in their recent paper have extended these analytical series-based approximations to be valid for the entire range of Z_R and δ_b. However, their use for practical

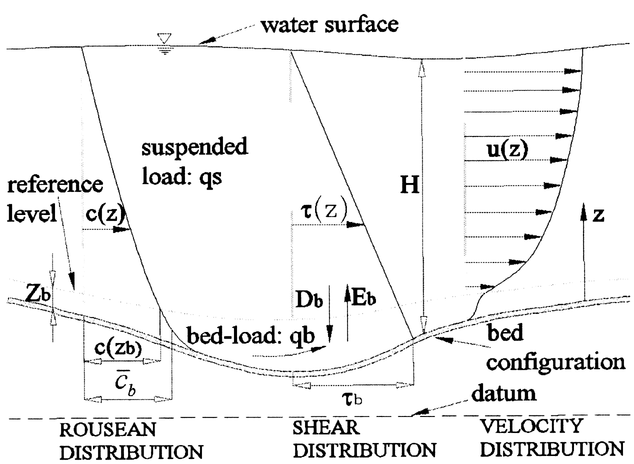

Fig. 1. Open-channel distribution

Table 1. Coefficients for INT_1 Formulation

δ_b	c_0^I	c_1^I	c_2^I	c_3^I	c_4^I	c_5^I	c_6^I	R^2
0.01	1.4852	0.2025	14.087	20.918	−10.91	2.034	−0.1345	1.00
0.02	1.2134	1.9542	10.613	6.0002	−3.6259	0.6938	−0.0462	1.00
0.03	1.1409	2.4266	8.2541	2.4058	−1.7617	0.3474	−0.0234	1.00
0.04	1.1138	2.5982	6.7187	1.029	−1.001	0.2045	0.0139	1.00
0.05	**1.1038**	**2.6626**	**5.6497**	**0.3822**	**−0.6174**	**0.1315**	**−0.0091**	**1.00**
0.06	1.102	2.6809	4.864	0.0422	−0.3989	0.0894	−0.0063	1.00
0.07	1.1048	2.6775	4.2624	−0.1487	−0.2639	0.0629	−0.0045	1.00
0.08	1.1104	2.6636	3.787	−0.2598	−0.1757	0.0454	−0.0033	1.00
0.09	1.1178	2.6448	3.4019	−0.3254	−0.1156	0.0333	−0.0025	1.00
0.1	1.1266	2.6239	3.0838	−0.3636	−0.0734	0.0246	−0.0019	1.00

Table 2. Coefficients for INT_2 Formulation

δ_b	c_0^{II}	c_1^{II}	c_2^{II}	c_3^{II}	c_4^{II}	c_5^{II}	c_6^{II}	R^2
0.01	1.151	2.1787	7.6572	−0.2777	−0.57	0.1424	−0.0105	1.00
0.02	1.1428	2.4442	4.2581	−0.4713	−0.1505	0.0467	−0.0036	1.00
0.03	1.1744	2.4172	3.0015	−0.4405	−0.049	0.0218	−0.0018	1.00
0.04	1.2143	2.364	2.3373	−0.3955	−0.0104	0.0116	−0.001	1.00
0.05	**1.2574**	**2.3159**	**1.9239**	**−0.3558**	**0.0075**	**0.0064**	**−0.0006**	**1.00**
0.06	1.3023	2.2773	1.6411	−0.3228	0.0167	0.0035	−0.0004	1.00
0.07	1.3486	2.2481	1.4351	−0.2955	0.0216	0.0017	−0.0003	1.00
0.08	1.3961	2.2269	1.2782	−0.2728	0.0243	0.0005	−0.0002	1.00
0.09	1.445	2.2125	1.1548	−0.2536	0.0258	−0.0002	−0.0001	1.00
0.1	1.4952	2.2041	1.0552	−0.2372	0.0265	−0.0008	−0.00005	1.00

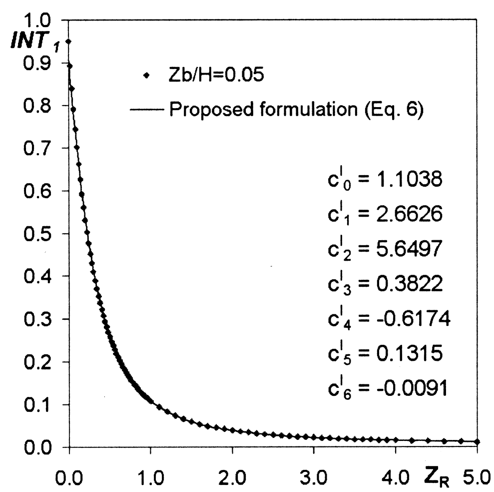

Fig. 2. Comparison between analytical solution and regression analysis for $INT_1(\delta_b=0.05)$

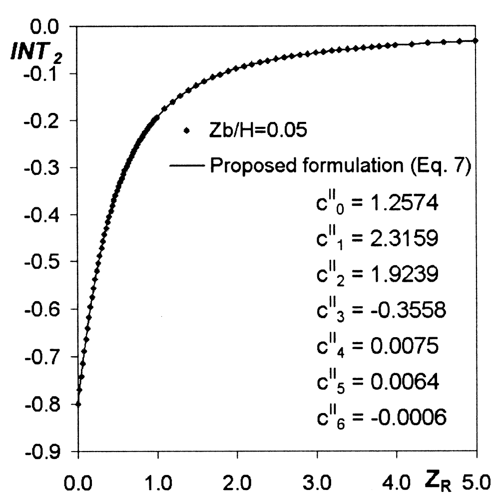

Fig. 3. Comparison between analytical solution and regression analysis for $INT_2(\delta_b=0.05)$

calculations is still limited since the computation of the integrals requires the implementation of a numerical subroutine or function. The effort and time required for incorporating any numerical subroutine or function for the direct integration of INT_1 and INT_2 suggest the necessity of developing more practical approaches that can be implemented more readily into a numerical model. The present discussion attempts to fulfill this last item.

Analysis and Formulations for Integrals INT_1 and INT_2

INT_1 and INT_2 are computed numerically (also computed using similar analytical approximations as proposed by the writers) for different Z_R and δ_b values. Then a sixth-order regression analysis of the results is performed in order to obtain simple formulations for INT_1 and INT_2, as shown by Eqs. (6) and (7) (see coefficients in Table 1 and Table 2):

$$INT_1 = \frac{1}{c_0^I + c_1^I Z_R + c_2^I Z_R^2 + c_3^I Z_R^3 + c_4^I Z_R^4 + c_5^I Z_R^5 + c_6^I Z_R^6} \quad (6)$$

$$INT_2 = \frac{-1}{c_0^{II} + c_1^{II} Z_R + c_2^{II} Z_R^2 + c_3^{II} Z_R^3 + c_4^{II} Z_R^4 + c_5^{II} Z_R^5 + c_6^{II} Z_R^6} \quad (7)$$

The writers have presented series-based approximations of the Einstein integrals. The discussers have developed additional formulations of these integrals with their practical application in mind. The proposed alternative approximations need some coefficients for different values of δ_b; however, in most existing formulas for estimating sediment entrainment or near-bed concentrations under equilibrium conditions, the reference level $\delta_b = z_b/H$ is taken to be 0.05 (Itakura and Kishi 1980; Celik and Rodi 1984; Akiyama and Fukushima 1986; García and Parker 1991, among others), which involves seven coefficients for each integral. Figs. 2 and 3 show comparisons of the exact solutions (dots) against the proposed practical formulation (continuous line) for INT_1 and INT_2, respectively. Good agreement was found by using these practical formulations.

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The discussers have found an explicit approximation for the Einstein Integrals which result in a discrepancy less than 1% to the approximation of the writers, though the approximation has greater errors when the dimensionless reference height becomes bigger than *0.01 for the first integral* and *0.001 for the second integral*. However, a huge fraction of natural conditions is covered by this approximation. The calculation speed is more than two orders faster than the approximation of the writers and nearly twice as fast as the recurrence formula suggested by Guo et al. (1996), which is the foundation for the suggested formula. The combination of the explicit approximation and the recurrence formula is suggested as an optimal algorithm for the estimation of both integrals. In the following, the recurrence formula of Guo et al. (Approximation II) and the explicit approximation (Approximation III) by the discusser will be compared with respect to performance and accuracy with the actual formulation of the writers (Approximation I). The derivation of the explicit approximation for the integral $J1$ starts with the identical idea as the writers', which is multiple application of the recurrence formula (1)

$$J1_{(\xi_b,z)} = \left(\frac{1}{z-1}\right) \cdot \left[\frac{(1-\xi_b)^z}{\xi_b^{z-1}}\right] - \left(\frac{1}{z-1}\right) \cdot J1_{(\xi_b,z-1)} \tag{1}$$

In Eq. (1) $\xi_b = a/h$, with a the reference height, h the water depth, and z the Rouse number. To get an explicit formula for the calculation of $J1$, Eq. (1) was expanded three times in z

$$J1_{(\xi_b,z)} = \frac{z \cdot \pi}{\sin[z \cdot \pi]} - \frac{\xi_b^{1-z}}{1-z} \tag{2}$$

The explicit approximation was obtained with the aid of Eq. (2), which is an approximation for $J1$ when $z<1$ (see Guo et al.). This equation leads to a discrepancy in the solution of the writers, which is less than 1% in if ξ_b is smaller than 0.01 (Fig. 1).

$$\begin{aligned} J1_{(\xi_b,z)} = {} & \left(\frac{1}{z-1}\right) \cdot \left[\frac{(1-\xi_b)^z}{\xi_b^{z-1}}\right] \\ & - \left(\frac{z}{z-1}\right) \cdot \left\{ \left(\frac{1}{z-2}\right) \cdot \left[\frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}}\right] \right. \\ & - \left(\frac{z-1}{z-2}\right) \cdot \left\{ \left(\frac{1}{z-3}\right) \cdot \left[\frac{(1-\xi_b)^{z-2}}{\xi_b^{z-3}}\right] \right. \\ & \left. \left. - \left(\frac{z-2}{z-3}\right) \cdot \left[\frac{(z-3) \cdot \pi}{\sin[(z-3) \cdot \pi]} - \frac{\xi_b^{4-z}}{4-z}\right] \right\} \right\} \end{aligned} \tag{3}$$

The explicit approximation of the second Einstein Integral is also based on the publication of Guo et al., but the accuracy is not as good as the approximation of the first Einstein Integral. The starting point for the derivation of the explicit approximation for $J2$ is again the recurrence formula given by Guo et al. for $J2$:

$$J2_{(\xi_b,z)} = \frac{1}{z-1} \cdot \left[\left(\frac{(1-\xi_b)^z}{\xi_b}\right) \cdot \ln(\xi_b) - z \cdot J2_{(\xi_b,z-1)} + J1_{(\xi_b,z)}\right] \tag{4}$$

The explicit approximation is achieved in a similar way as for $J1$. Guo et al. have suggested for $z<1$ the following approximation for $J2$:

$$J2_{(\xi_b,z)} = -\frac{z \cdot \pi \cdot f_{(z)}}{\sin(z \cdot \pi)} - \frac{\xi_b^{z-1}}{1-z} \cdot \log(\xi_b) + \frac{\xi_b^{z-1}}{(1-z)^2}$$

$$f_{(z)} = (1-\gamma) - \log(2-z) + \frac{1}{1-z} + \frac{1}{2 \cdot (1-z)} + \frac{1}{24 \cdot (2-z)^2} \tag{5}$$

Here $f(z)$=approximation of a more complicated functional in terms of Gamma functions given by Guo and Wood (1995). An explicit approximation for $J2$ was found after double expansion of Eq. (4) and replacement of the terms $J1$ and $J2$ in Eq. (6), with the functions summarized in Eq. (7). The final equation shows a discrepancy to the solution of the writers which is less than 1% for a reference height that is smaller than 0.001 (Fig. 2)

$$\begin{aligned} J2_{(\xi_b,z)} = {} & \left(\frac{1}{z-1}\right) \cdot \left\{ \log(\xi_b) \cdot \frac{(1-\xi_b)^z}{\xi_b^{z-1}} \right. \\ & - z \cdot \left[\left(\frac{1}{z-2}\right) \cdot \left(\log(\xi_b) \cdot \frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}} \right. \right. \\ & \left. \left. \left. - (z-1) \cdot J2_{(\xi_b,z-3)} \cdot J1_{(\xi_b,z-2)} \right) \right] + J1_{(\xi_b,z)} \right\} \end{aligned} \tag{6}$$

The approximation of the function $f_{(z)}$ is not straightforward as the second term becomes undefined when the argument is negative ($z>4$). Numerical experiments with (6) have shown that a good approximation can be achieved if the absolute argument in the log function is calculated.

$$J2_{(\xi_b,z-3)} = -\frac{(z-2) \cdot \pi \cdot f_{(z)}}{\sin((z-2) \cdot \pi)} - \frac{\xi_b^{3-z}}{3-z} \cdot \log(\xi_b) + \frac{\xi_b^{3-z}}{(3-z)^2}$$

$$\begin{aligned} J1_{(\xi_b,z-2)} = {} & \left(\frac{1}{z-2}\right) \cdot \left[\frac{(1-\xi_b)^{z-1}}{\xi_b^{z-2}}\right] \\ & - \left(\frac{z-1}{z-2}\right) \cdot \left[\frac{(z-2) \cdot \pi}{\sin[(z-2) \cdot \pi]} - \frac{\xi_b^{3-z}}{3-z}\right] \end{aligned}$$

$$f_{(z)} = (1-\gamma) - \log(|4-z|) + \frac{1}{3-z} + \frac{1}{2 \cdot (4-z)} + \frac{1}{24 \cdot (4-z)^2} \tag{7}$$

The results of the integrals $J1$ and $J2$ obtained with the approximation of the writers have been compared with the recurrence formula of Guo et al. and the explicit approximation suggested by the discussers

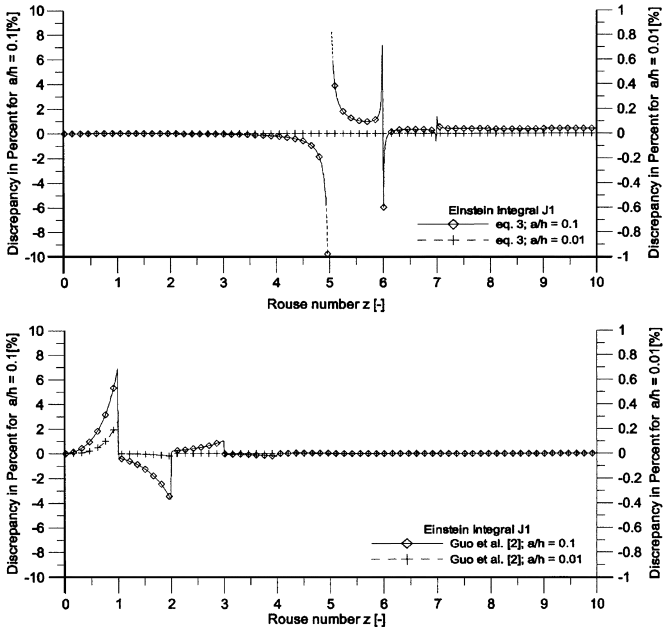

Fig. 1. Discrepancy in percent between the suggested explicit approximation (top) and the recurrence formula [2] (bottom) in comparison to the approximation of the writers for $J1$. Discontinues lines indicates that the curve is limited to the range of the axis.

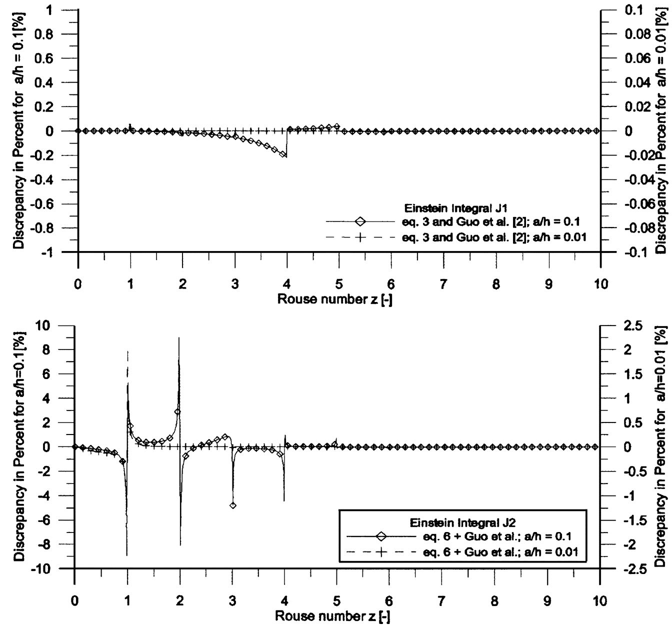

Fig. 3. Error in percent between the suggested optimal approximation algorithm and the results of the writers

$$z = \{z \in (0.01, 0.02, \ldots 10.0)\}$$
$$\xi_b = \{\xi_b \in (10^{-5.0}, 10^{-4.9}, \ldots 10^{-1.0})\} \quad (8)$$

Both approaches lead to rather big errors, if the Rouse number z approaches integer values and if the dimensionless reference height ξ_b approaches unity. When the reference height a becomes small in comparison to h, the discrepancies disappear. The reason is that the recurrence formulas have been derived by Guo et al. under the assumption that a is small in comparison to h.

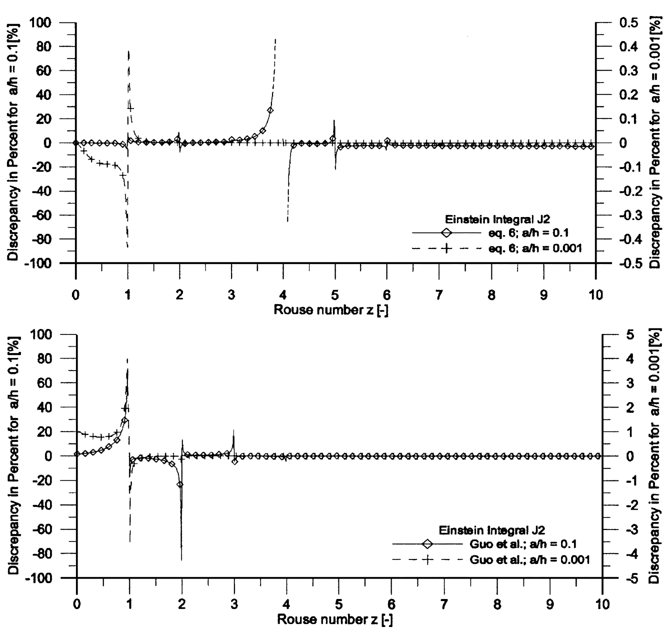

Fig. 2. Discrepancy in percent between the suggested explicit approximation (top) and the recurrence formula [2] (bottom) in comparison to the approximation of the writers for $J1$

To achieve optimal performance and accuracy, the recurrence formula and the explicit approximation can be applied for the distinct areas in the co-domain in which these algorithms result in small errors in comparison to the exact solution. In Fig. 3, the results for such an algorithm are presented. It can be seen that for the integral $J1$ the discrepancy reduces to less than 1% for $J1$. For $J2$ the discrepancies are still one order higher than for the integral $J1$, though significantly reduced in comparison to Fig. 2. The error for $J2$ growths are maximal for $z=1$ and $\xi_b>0.01$, but it is still less than 2.5% while $\xi_b<0.01$. The most accurate approximation of the writers can be used if higher accuracy is anticipated in the designated regions.

We have summarized the computation time for the calculation of 10 times the co-domain [Eq. (8) for these three different approximations (see Fig. 4). The approximation of the writers is the slowest one due to its numerical complexity, though it is the most exact one. The explicit approximation is nearly twice as fast as the recurrence formula and more than two orders faster than the

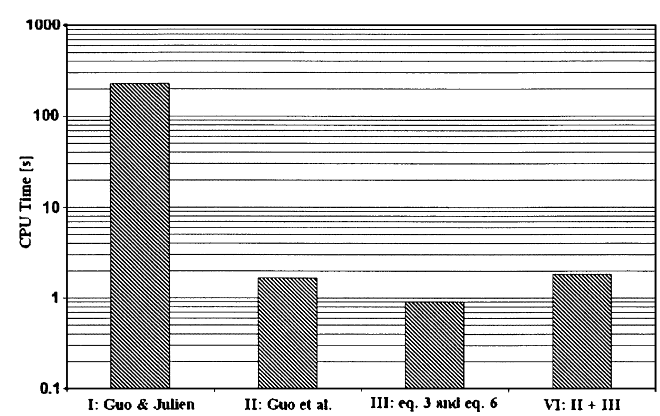

Fig. 4. Calculation time of the various approximations for the calculation of 10 times the co-domain

approximation of the writers. The combination of Approximations II and III is still more than two orders faster than the recent one suggestd by the writers.

Conclusions

An optimal scheme for calculating the Einstein Integrals has been proposed on the foundation of the work of Guo et al. and the suggested explicit approximation. The new suggested algorithm can be efficiently used in morphodynamic models like TIMOR (TIdal MORphodynamics; Zanke 2002), which are based on the solution of vertical integrated flow models. In such kinds of applications the suspended sediment transport rates have to be calculated for every time-step, grid point, and grain size, which leads to high calculation times. Therefore the improvement of the calculation speed and accuracy has direct impact on the performance of the morphodynamic model. This study was only possible due to the work of Prof. Junke Guo in the past 10 years concerning the derivation of an approximation for the Einstein Integrals. In the latest publication, discussed here, the writers found the most exact approximation of these equations to date, which is considered to be an important step in science.

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The writers have proposed infinite series expressions for evaluating the Einstein integrals. The discusser would like to mention the following points regarding the note:

1. Eq. (3) is only valid for $z<1$ because for $z\geqslant 1$ both the integrals on the right-hand side are *divergent*. Similarly, Eqs. (7) and (8) are valid only for $z<1$. While this limitation was stated explicitly before Eqs. (5) and (9), it should have been mentioned at other relevant places also.
2. While the range of z has been stated $(z<10)$ in the note, that for E has not been mentioned. The value of E used by the writers ranges from 0.00001 to 0.1. Obviously, if E is higher than 0.1, the convergence of the series in Eqs. (9) and (18) would be quite slow. In fact, if $E>0.5$, Eq. (8) will not be valid since $F_1(-\infty)\neq 0$ for $E/(1-E)>1$. Such high values of E may be practically impossible, but mathematical rigor demands a passing mention of the fact. The sentence in the Conclusions section stating, "…*valid over the entire range of the Rouse number z and the relative bed-layer thickness E*," should therefore be qualified with additional information.
3. While integer (and $n+1/2$) values of z may be useful for comparing the proposed algorithm with exact solutions, it is very unlikely that the actual value of z would be an *exact* integer. The discusser feels that the paragraph following Eq. (13) is therefore not really appropriate. In fact, it may be more desirable to have a single expression [Eq. (9)] for all values of z with the provision that if z is an exact integer, its value would be taken as $z\pm 0.001$!
4. In the paragraph above Eq. (20), rapid convergence of series (8) and (17) should be based on rapid approach to zero of $[E/(1-E)]^{k-z}$, and not E^{k-z}. As discussed in point 2 above, if $E>0.5$, E^{k-z} will still approach zero rapidly but the series will not converge.
5. While the writers mention the limit $z=10$ before Eq. (20), the maximum relative error for the approximation is stated only for $z<6$. It was found that the error increases with further increase in z and becomes more than 0.8% for $z=10$. While this may be acceptable for practical purposes, the approximation may be considerably improved by considering the limiting behavior of the series in Eq. (20). The following approximation was derived with the use of the limiting behavior and was found to have a maximum error of only 0.03% over the entire range $z>0$:

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{z+k}\right)=\ln(1+1.781z)-\frac{0.1361z}{(1+1.284z)^{2.150}} \quad (1)$$

6. An alternative methodology for deriving series expressions for J_1 is described below: Since $(1-\xi)/\xi$ is less than 1 for $\xi>0.5$ and more than 1 for $\xi<0.5$, we write

$$J_1(z,E)=\int_E^{0.5}\left(\frac{1-\xi}{\xi}\right)^z d\xi+\int_{0.5}^{1}\left(\frac{1-\xi}{\xi}\right)^z d\xi \quad (2)$$

Making the substitution $x=\xi/(1-\xi)$ in the first integral and $x=(1-\xi)/\xi$ in the second integral, we get

$$J_1(z,E)=\int_{E_*}^{1}x^{-z}(1+x)^{-2}dx+\int_0^1 x^z(1+x)^{-2}dx$$
$$=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}i[1-E_*^{i-z}]}{i-z}+\sum_{i=1}^{\infty}\frac{(-1)^{i-1}i}{i+z} \quad (3)$$

where $E_*=E/(1-E)$. Both these series, however, converge slowly. Also, for integer z values, the computations have to make use of the fact that $\lim_{p\to 0}(x^p-1)/p=\ln x$. To avoid the slow convergence, the following approximation was obtained with a maximum error of 0.18% using the data for $0\leqslant z\leqslant 5$ (in steps of 0.5) and $-5\leqslant \log E\leqslant -1$ (in steps of 1):

$$J_1(z,E)=-\frac{E_*^{1-z}-1}{1-z}+2.061\frac{E_*^{2-z}-1}{2-z}-1.385\frac{E_*^{2.6-z}-1}{2.6-z}+\frac{0.3327}{0.6703+z} \quad (4)$$

Similarly, $J_2(z,E)$ was approximated with a maximum error of 0.03% as

$$J_2(z,E) = \frac{E_*^{1-z}[1-(1-z)\ln E_*]-1}{(1-z)^2} - 1.903\frac{E_*^{2-z}[1-(2-z)\ln E_*]-1}{(2-z)^2} + 2.022\frac{E_*^{2.6-z}[1-(2.6-z)\ln E_*]-1}{(2.6-z)^2} - \frac{0.2914}{1.652+z} \tag{5}$$

Again, in the unlikely event of z being exactly equal to 1, 2, or 2.6, we make use of the fact that

$$\lim_{p\to 0} \frac{x^p[1-p\ln x]-1}{p^2} = -\frac{(\ln x)^2}{2}$$

The discusser feels that Eqs. (4) and (5) above would perform better than the writers' Eqs. (9) and (18) for computation of the Einstein integrals.

Closure to "Efficient Algorithm for Computing Einstein Integrals" by Junke Guo and Pierre Y. Julien

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We would like to thank all the discussers for their constructive comments and for providing alternative approximations of the Einstein integrals. It is commonly agreed that these two integrals remain elusive to exact solutions. However, a lot of progress has been made in developing fast and accurate approximations. Abad and Garcia emphasize the need for simpler algorithms and propose a polynomial approximation. Srivastava provides keen insight into the convergence of the series and offers improvements. Roland et al. present an error analysis as well as a valuable comparison of the computational time of various algorithms. Besides polynomial approximations and different series expansions, other e-mail communications by A. R. Kacimov pointed to the possible use of hypergeometric series as well as software packages like *MatLab*, *Maple*, and *Mathematica*. Four main issues are raised in the discussions and they are addressed in the following sequence: (1) convergence; (2) algorithm efficiency; (3) accuracy; and (4) computational time. This closure also includes a comparative analysis of the different algorithms and an application example on the Missouri River.

First, regarding convergence, Srivastava correctly points out that the proposed series are not convergent when the bed layer thickness E is greater than 0.5. Well, this is a trivial case because when $E>0.5$, sediment is exclusively transported as bedload and the integration is not required. For most practical applications, the value of E is relatively small. For a typical laboratory flume experiment, given a grain diameter of $D=1$ mm and a flow depth of $h=10$ cm, the value of E is 0.02. The value of E for field applications is even smaller, as per the example below.

Second, all the discussions express concerns about the computational efficiency of the proposed method. It was readily acknowledged in the Technical Note that the first series of J_2 in Eq. (18) converges slowly. Calculation times of the order of a second were not considered excessive, and accuracy was preferred over CPU time. In view of the discussions, however, it is now possible to substantially improve computational efficiency. The results of an earlier formulation proposed by Guo and Wood can be expanded and simplified to replace the former Eq. (14) with

$$\begin{aligned}\int_0^1 \left(\frac{1-\xi}{\xi}\right)^z \ln\xi d\xi &= -\frac{z\pi}{\sin z\pi}[(1-\gamma)-\psi(1-z)] \\ &= -\frac{z\pi}{\sin z\pi}\left[(1-\gamma)-\psi(n+1-z)+\sum_{k=1}^{n}\frac{1}{k-z}\right] \\ &= -\frac{z\pi}{\sin z\pi}\left[(1-\gamma)-\ln(n-z)-\frac{1}{2(n-z)}+\frac{1}{12(n-z)^2}\right. \\ &\quad \left. -\frac{1}{120(n-z)^4}+\cdots+\sum_{k=1}^{n}\frac{1}{k-z}\right]\end{aligned} \tag{1}$$

in which $\gamma=0.5772156649015328606065\ldots$ and $\psi(x)$ is the psi function, a special function. For calculations, it is suggested that n take as $n=\text{ceil}(z)+2$, in which ceil(z) means the ceiling of the value of z. For example, if $z=0.1$, $n=\text{ceil}(0.1)+2=3$; if $z=3.7$, $n=\text{ceil}(3.7)+2=6$. Although it is derived for $z<1$, Eq. (1) above is valid for any noninteger value of z. One can demonstrate that Eq. (1) requires fewer terms than Eq. (14) and converges rapidly.

After replacing Eq. (14) with Eq. (1), the proposed algorithm can be slightly modified. For noninteger z, i.e., $|z-\text{round}(z)|>0.005$,

- *Step 1*: Estimate $F_1(z)$ from Eq. (8) using a maximum $k=\text{ceil}(z)+4$.
- *Step 2*: Estimate $J_1(z)$ from Eq. (9).
- *Step 3*: Estimate Eq. (1) above using a value $n=\text{ceil}(z)+2$.
- *Step 4*: Estimate $F_2(z)$ from Eq. (17) using a maximum $k=\text{ceil}(z)+2$.
- *Step 5*: The value of $J_2(z)$ is then obtained by subtracting the result of *Step 4* from the result of *Step 3*.

For integer z values, one can use the same steps as those in the technical note or a recurrent formula like Eq. (10). A program in both *FORTRAN* (einstein.f90) and *MatLab* (einstein.m) for the above algorithm, together with alternatives proposed by the discussers, can be downloaded from ⟨http://myweb.unomaha.edu/~junkeguo⟩ or ⟨http://www.engr.colostate.edu/%7Epierre/ce_old/Projects/index.html⟩. The software programs of readers interested in sharing their source codes will be made available at the same site.

Third, in terms of accuracy, the formulation of Roland et al. is a combination of the three versions of Guo and Wood, Guo et al., and Guo and Julien. There is, however, some concern regarding their statement "the error for J_2 becomes maximal for $z=1$ and E or $\xi_b>0.01$ but it is still less than 10%." Values of $z=1$ are not unusual in practice and 10% looks like a disquietly large error. Roland kindly provided a *FORTRAN* code of his algorithm, but we experienced difficulties running his software and replicating his results. A comparison of several algorithms is presented in this section. Values of $E=0.1$ and $z=0.55$, 1.55, 2.55, 3.55, and 4.55 are considered for the calculations. This large value for E is se-

Table 1. Comparison of Various Algorithms at $E=0.1$

Method	$z=0.55$	$z=1.55$	$z=2.55$	$z=3.55$	$z=4.55$
		(a) Value of J_1			
Exact value	0.97458	2.7326	13.002	77.623	519.35
Guo and Julien (Technical Note)	0.97458	2.7326	13.002	77.623	519.35
Guo and Julien Eq. (1)	0.97458	2.7326	13.002	77.623	519.35
Abad and Garcia	0.97433	2.7350	12.997	77.620	519.19
Srivastava	0.97362	2.7313	12.982	77.604	519.81
		(b) Value of J_2			
Exact value	−1.1201	−4.4912	−24.513	−155.85	−1078.9
Guo and Julien (Technical Note)	−1.1201	−4.4913	−24.513	−155.85	−1078.9
Guo and Julien Eq. (1)	−1.1201	−4.4913	−24.513	−155.85	−1078.9
Abad and Garcia	−1.1200	−4.4921	−24.513	−155.84	−1078.0
Srivastava	−1.1258	−4.5535	**−28.742**	−155.93	−1107.8

Note: The exact values are numerical integrations with Maple.

lected here because Roland et al. showed that it corresponds to larger differences between approximations and exact solutions. The results obtained for the two integrals J_1 and J_2 using four different algorithms (from the Technical Note; Eq. (1); Abad and Garcia; and Srivastava) are summarized in Table 1. They are compared with the exact values obtained from numerical integration in Maple. It is concluded that like the results from the original note, the improvement with Eq. (1) is accurate. Other methods also produce fine results. However, the method of Srivastava can generate errors larger than the value of 0.03% cited in his discussion. For instance, Table 1 shows a 17% error for J_2 at $z=2.55$. Things get worse around $z=2.6$ with errors as large as 800% when $z=2.6-0.001=2.599$. A positive value for $J_2=186.19$ is also incorrectly obtained when $z=2.6+0.001=2.601$.

Fourth, regarding computational time, the CPU times of three algorithms are considered. The calculations were performed with *MatLab* and *FORTRAN* on a Dell Notebook. The results of the methods are very comparable at about 0.015625 s. In addition, one can produce Figs. 1 and 2 in the Technical Note in only 0.875 s. It is fair to say that these algorithms to solve the two Einstein integrals are fast and accurate. Further comparison of the algorithms is presented in the following field application to the Missouri River.

Example: A calculation example with field data is presented here. The site of the Missouri River near Omaha, Neb., is selected for sediment transport calculations using Einstein's method. The main parameters are slope $S=0.00012$, flow depth $h=7.8$ ft$=2.38$ m, and water temperature $T=7°$C. The measured velocity profile u and suspended sand concentration c for the fraction passing a 0.105 mm sieve and retained on a 0.074 mm sieve are shown in Table 2 (from Julien 1995, p. 202). The assignment is to calculate the unit sediment discharge for this size fraction.

Solution: Several algorithms are used for the determination of J_1 and J_2. We use the methods of Srivastava, Abad and Garcia, and Eq. (1) for comparisons with the exact solution. The measured velocity profile and the suspended sediment concentration distribution are fitted as follows:

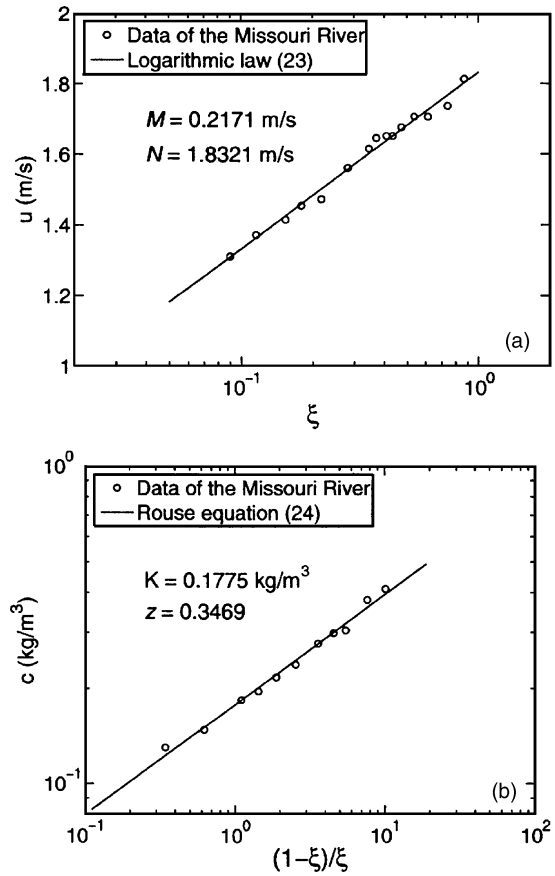

Fig. 1. Curve-fittings of velocity and concentration distributions

Table 2. Measurements of Distributions of Velocity and Concentration in Missouri River

ξ	u (m/s)	c (kg/m^3)
0.090	1.31	0.411
0.115	1.37	0.380
0.154	1.41	0.305
0.179	1.45	0.299
0.218	1.47	0.277
0.282	1.56	0.238
0.346	1.62	0.217
0.372	1.65	—
0.410	1.65	0.196
0.436	1.65	—
0.474	1.68	0.184
0.538	1.71	—
0.615	1.71	0.148
0.744	1.74	0.130
0.872	1.81	—
1.000	—	—

$$u = M \ln \xi + N \tag{2}$$

$$c = K\left(\frac{1-\xi}{\xi}\right)^z \tag{3}$$

in which ξ=distance relative to the flow depth from the bed; and M, N, K, and z=fitting constants from the measurements. Fig. 1 gives $M=0.2171$ m/s, $N=1.8321$ m/s, $K=0.1775$ kg/m^3, and $z=0.3469$. With Eqs. (2) and (3), the Einstein bed-load function can be written as

$$q_T = KMh\left[4.61\frac{(1-E)^z}{E^{z-1}} + \frac{N}{M}J_1(z,E) + J_2(z,E)\right] \tag{4}$$

in which q_T=unit sediment discharge including bed load, the 1st term, and suspended load, the 2nd and 3rd terms. Given the median sediment size $d_m=(0.105+0.074)/2=0.0895$ mm, the bed-layer thickness is then $a=2d_m=0.179$ mm, and the relative thickness $E=a/h=7.5\times10^{-5}$. Compared with the exact value of 0.7520 kg/s.m from Maple, the improved algorithm with Eq. (1) gives $q_T=0.7520$ kg/s.m with a CPU time 0.015 s; and Srivastava's method gives $q_T=0.7496$ kg/s.m, also with a CPU time 0.015 s. The method of Abad and Garcia with the coefficients at the lowest value $E=\delta_b=0.01$ gives $q_T=4.3667$ kg/s.m with the same CPU time. This example shows that Eq. (1) and Srivastava's method are comparable. The method of Abad and Garcia leads to a 580% error in this case because of the small values of E. It is clear that a better approximation of the 14 coefficients would improve the method of Abad and Garcia when $\delta_b<0.01$. The *FORTRAN* program (missouri.f90) and *MatLab* program (missouri.m) for this example are available at the previously mentioned Web site.

In summary, the insightful discussions lead us to a substantially improved solution for J_2 in Eq. (1). This algorithm is quick, accurate, efficient, and convergent. The other methods proposed in the discussions are also quite good although some deficiencies are observed. For instance, the method of Abad and Garcia can lead to substantial errors when $E<0.01$ or $\delta_b<0.01$. The method of Srivastava is also prone to large errors around $z=2.6$. Software and algorithms can be downloaded and shared at the writers' Web sites. In view of such excellent discussions of this article, H. A. Einstein would certainly appreciate the renewed interest in the use of his method.

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Discussion of "Simulation of Flow and Mass Dispersion in Meandering Channels" by Jennifer G. Duan

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This paper is of great interest to the community of hydraulic engineering. The writer has developed an enhanced two-dimensional numerical model for simulating flow hydrodynamics and mass transport in meandering channels. The dispersion terms in momentum equations play an important role in the writer's model. The dispersion terms result from the discrepancy between the depth-averaged velocity and the actual velocity. The writer used the logarithmic law to describe the streamwise velocity profile

$$\frac{u_l}{u_*} = \frac{1}{\kappa}\ln\left(\frac{z}{z_0}\right) \tag{1}$$

where u_l=streamwise velocity; u_*=shear velocity; κ=Von Kármán's constant (=0.4); z=vertical coordinate; and z_0=constant (a certain distance from the wall).

Zero-Velocity Level

The logarithmic velocity profile shows that the flow has zero velocity at $z=z_0$ and obviously requires a condition of $z\geqslant z_0$. The constant z_0, also called the zero-velocity level, is of the same order of magnitude as the viscous sublayer thickness and is a function of whether the boundary is hydraulically smooth or rough.

The constant z_0 is an important parameter in the dispersion terms of the momentum equations, as shown in Eqs. (15) to (17) of the original paper. To calculate z_0, the writer used Eq. (2), which contains three relations for flow in three boundary regimes, namely, those of the hydraulically smooth regime ($\mathsf{R}_e\leqslant5$), the completely rough regime ($\mathsf{R}_e\geqslant70$), and the transitional regime ($5<\mathsf{R}_e<70$), according to the magnitude of roughness Reynolds number R_e, defined as $\mathsf{R}_e=u_*k_s/\nu$, where ν=kinetic viscosity and k_s=roughness height

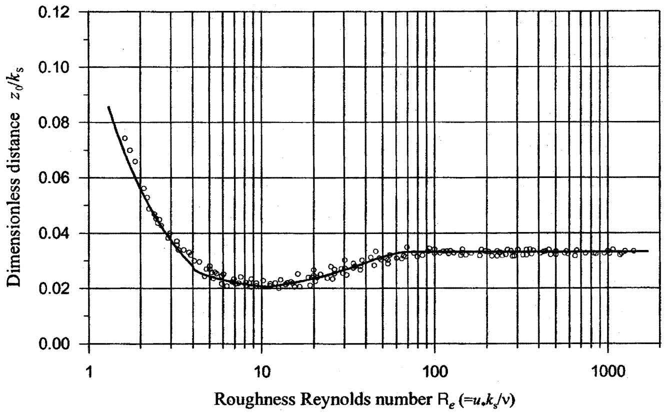

Fig. 1. Ratio of z_0/k_s in terms of roughness Reynolds number R_e

$$z_0 = 0.11\frac{\nu}{u_*}, \quad \frac{u_* k_s}{\nu} \leqslant 5$$

$$z_0 = 0.033 k_s, \quad \frac{u_* k_s}{\nu} > 70 \tag{2}$$

$$z_0 = 0.11\frac{\nu}{u_*} + 0.033 k_s, \quad 5 < \frac{u_* k_s}{\nu} < 70$$

The first two relations in Eq. (2) are the well-known empirical relations of z_0 for flows over hydraulically smooth and completely rough boundaries, respectively. However, there is a lack of relation of z_0 in an explicit form for flow in a transitional regime. The writer has proposed an explicit relation of z_0 for the transitional regime in the paper, as shown in Eq. (2). Unfortunately, the proposed relation of z_0 by the writer for the transitional regime is improper.

For convenience in the present discussion, the relations of z_0 in Eq. (2) are rearranged in terms of z_0/k_s and R_e, and renamed as follows:

$$\frac{z_0}{k_s} = \frac{0.11}{R_e}, \quad \text{for } R_e \leqslant 5 \tag{3a}$$

$$\frac{z_0}{k_s} = \frac{0.11}{R_e} + 0.033, \quad \text{for } 5 < R_e < 70 \tag{3b}$$

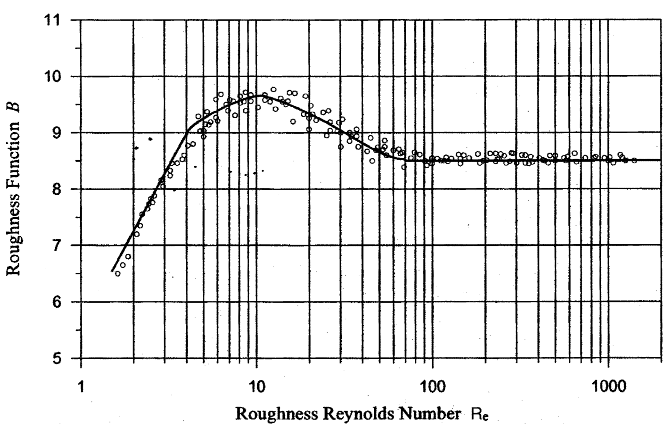

Fig. 2. Roughness function B in terms of roughness Reynolds number R_e; solid line plotted according to Eqs. (6*a*)–(6*d*)

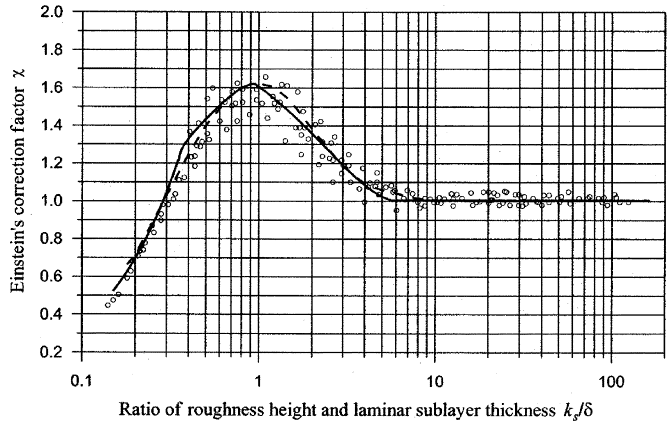

Fig. 3. Correction factor χ in terms of k_s/δ; solid line was plotted according to Eqs. 8(*a*)–(*d*), whereas dashed line is Einstein's curve

$$\frac{z_0}{k_s} = 0.033, \quad \text{for } R_e > 70 \tag{3c}$$

Obviously, the value of z_0/k_s for $R_e \to 70$ from Eq. (3*b*) is close to that from Eq. (3*c*) but the value of z_0/k_s for $R_e \to 5$ from Eq. (3*b*) is much larger (2.5 times larger) than that from Eq. (3*a*). The values of z_0 calculated from the writer's proposed relation are significantly overestimated for flow in a transitional regime. Therefore, improving the discrepancy is necessary.

After reviewing Nikuradse's experimental data, shown in Fig. 20.21 of Schlichting's book (Schlichting, 1979, p. 620), and slightly adjusting the limit between the smooth and transitional regimes, the discusser proposes the following empirical relations of z_0/k_s for flow in hydraulically smooth ($R_e \leqslant 4$), transitional ($4 < R_e < 70$), and completely rough ($R_e \geqslant 70$) regimes

$$\frac{z_0}{k_s} = \frac{0.11}{R_e}, \quad \text{for } R_e \leqslant 4 \tag{4a}$$

$$\frac{z_0}{k_s} = 0.0275 - 0.007\sqrt{\sin\left(\frac{R_e - 4}{14}\right)\pi}, \quad \text{for } 4 < R_e \leqslant 11 \tag{4b}$$

$$\frac{z_0}{k_s} = 0.0205 + \frac{0.0125}{\sqrt{2}}\sqrt{1 + \sin\left(\frac{R_e - 40.5}{59}\right)\pi}, \quad \text{for } 11 < R_e < 70 \tag{4c}$$

$$\frac{z_0}{k_s} = 0.033, \quad \text{for } R_e \geqslant 70 \tag{4d}$$

The relations of z_0/k_s in terms of R_e proposed by the discusser are close to Nikuradse's experimental data, with a discrepancy of less than 10%, as shown in Fig. 1.

Roughness Function *B* and Correction Factor χ

The logarithmic law of velocity distribution is also commonly written in the following form

$$\frac{u_l}{u_*} = 2.5 \ln\left(\frac{z}{k_s}\right) + B \tag{5}$$

where the constant B, called the roughness function, has the relation $B = -2.5 \ln(z_0/k_s)$; therefore, it yields that

$$B = -2.5 \ln\left(\frac{0.11}{\mathsf{R}_e}\right), \quad \text{for } \mathsf{R}_e \leqslant 4 \tag{6a}$$

$$B = -2.5 \ln\left(0.0275 - 0.007\sqrt{\sin\left(\frac{\mathsf{R}_e - 4}{14}\right)\pi}\right), \quad \text{for } 4 < \mathsf{R}_e \leqslant 11 \tag{6b}$$

$$B = -2.5 \ln\left(0.0205 + \frac{0.0125}{\sqrt{2}}\sqrt{1 + \sin\left(\frac{\mathsf{R}_e - 40.5}{59}\right)\pi}\right), \quad \text{for } 11 < \mathsf{R}_e < 70 \tag{6c}$$

$$B = 8.5, \quad \text{for } \mathsf{R}_e \geqslant 70 \tag{6d}$$

The values of B calculated from Eqs. (6*a*) and (6*d*) are also close to Nikuradse's experimental data, as shown in Fig. 2.

A logarithmic velocity distribution with a correction factor χ proposed by Einstein (1950) is given as

$$\frac{u_l}{u_*} = 5.75 \log\left(30.2\chi\frac{z}{k_s}\right) \tag{7}$$

The correction factor χ has a relation with z_0/k_s as $\chi = 0.0331(z_0/k_s)^{-1}$. Einstein (1950) proposed a well-known plot of χ against k_s/δ which is the ratio of the roughness height k_s and the thickness of laminar sublayer δ. The values of δ can be estimated by $\delta = 11.6\nu/u_*$, which yields that $k_s/\delta \approx 0.086\ \mathsf{R}_e$. Therefore, the empirical relations of χ against k_s/δ in various regimes can be written as follows:

$$\chi = 3.48\frac{k_s}{\delta}, \quad \text{for } \frac{k_s}{\delta} \leqslant 0.35 \tag{8a}$$

$$\chi = \frac{1}{0.83 - 0.21\sqrt{\sin\left(\frac{(k_s/\delta) - 0.35}{1.20}\right)\pi}}, \quad \text{for } 0.35 < \frac{k_s}{\delta} \leqslant 0.95 \tag{8b}$$

$$\chi = \frac{1}{0.62 + \frac{0.38}{\sqrt{2}}\sqrt{1 + \sin\left(\frac{(k_s/\delta) - 3.49}{5.08}\right)\pi}}, \quad \text{for } 0.95 < \frac{k_s}{\delta} < 6.03 \tag{8c}$$

$$\chi = 1.0, \quad \text{for } \frac{k_s}{\delta} \geqslant 6.03 \tag{8d}$$

Fig. 3 shows that the proposed relations of χ in terms of k_s/δ by the discusser are also close to Einstein's curve, as well as to Nikuradse's experimental data.

The empirical roughness-related relations for the ratio z_0/k_s, the roughness function B, and the correction factor χ against R_e (or k_s/δ) have been proposed in the present discussion. The proposed empirical relations are all in an explicit form and allow the writer and others to directly calculate the values of z_0/k_s (or B and χ) according to the values of R_e (or k_s/δ).

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Closure to "Simulation of Flow and Mass Dispersion in Meandering Channels" by Jennifer G. Duan

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The writer thanks the discusser for his comments. The following are responses to them.

Eq. (9) of the original paper is typically adopted by many textbooks for undergraduate students studying fluid dynamics (e.g. Van Rijn 1989). The experimental cases have a roughness Reynolds number greater than 60, which validates Eqs. (9*b*) and (9*c*) of the original paper. Since flows in all three cases are turbulent, Eq. (9*a*) for laminar flows has never been applied in the simulations. The developed model aims to simulate turbulent rather than laminar flow, because flows in laboratory experimental flume and natural rivers usually are turbulent, with a Reynolds number higher than 2,000 (Duan and Julien 2005). Calculating the roughness factor for laminar flows by using Eq. (9*a*) has no effect on the simulated results presented in the paper. For turbulent open-channel flow, the velocity profile has three distinct regions: the inner wall, the buffer, and the outer wall. The velocity profile can be expressed as (Nezu 2005)

$$\begin{cases} U^+ = y^+ & y^+ < 5 \\ U^+ = \frac{1}{\kappa}\text{In } y^+ + A + w(\xi) \text{ with } w(\xi) = \frac{2\Pi}{\kappa}\sin^2\left(\frac{\pi}{2}\xi\right) & y^+ > 11.6 \end{cases} \tag{44}$$

where $U^+ = U/U_*$; $y^+ = y/(\nu/U_*)$; A = roughness constant; $\xi = y/h$; h = flow depth; κ = Von Kármán constant; and Π = Coles's wake strength parameter (Coles 1956). An analytical solution for the buffer layer between the viscous sublayer and the log-law wake layer is not available in an analytical formula.

The roughness height for flow in laminar, transition, and turbulent in Eqs. 4(*a*) to 4(*d*) in the discussion matched Nikuradse's experimental data very well. However, Einstein's (1950) equation is for the velocity profile with a mobile bed of ripples or dunes bed forms. Nikuradse's experimental data is obtained at a fixed bed experiment for $\mathsf{R}_e < 1{,}200$, therefore, Eq. (7) of the discussion, derived by using Einstein (1950) and Nikuradse's experimental data, is only applicable when the roughness Reynolds number is less than 1,200. Additional experimental and field data (e.g., Tsujimoto et al. 1991; Papanicolaou and Hilldale 2002) need to be included to demonstrate the applicability of Eq. (7) to a fully turbulent flow with $\mathsf{R}_e > 1{,}200$.

In summary, Eq. (7) can be used to replace Eq. (9) of the original paper in calculating the roughness factor. Further research is needed to include more recent experimental data and extend the applicability of Eq. (7) to turbulent flows of high R_e numbers.

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